Topics in Ordered Semigroups

by

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Abstract

According to Krull, the combined study of an algebraic structure and an order structure goes back to the theory of relations developed by Eudoxus. The study was set forth by Euclid in the fifth book of his 'Elements'. (In 1966, Beckmann showed that the account in the fifth book is, in fact, concerned with additive fully ordered (f.o.) Archimedean semigroup). The development of the abstract theory of ordered algebraic system was then put forward by Huntington in 1902. He also studied f.o. semigroups and, in the first instance, f.o. groups. Fully ordered groups were originally thought of as being embedded in groups. Later on, Chehata and Vinogradov simultaneously and independently proved that there exist f.o. groups that cannot be embedded in groups. In the 1940's and 50's, Alimov, Tamari, Khion, Clifford and Chehata began a study of f.o. semigroups which are not necessarily embedded in groups. In the 1960's, Nakada, Dubreil-Jacotin and others worked on partially ordered semigroups. From that time the foundation of the theory of partially ordered semigroups has been well established and a very large number of publications and fruitful results have appeared. In 1963, the book "partially ordered algebraic systems", written by L. Fuchs was published. this was the first book to give a systematic presentation of all the past results obtained in partially (or fully) ordered algebraic systems. At the present time, work on f.o. semigroups is appreciably influenced by the work of Tamura and Saito. The French research school in the field of fully and partially ordered semigroups has been led for many years by Dubriel-Jacotin. Soviet Mathematicians also make a lot of contributions to the development of the theory of partially ordered semigroups.

As an object of study, f.o. and p.o. semigroups are of interests because of their applications in other branches of mathematics as well as in other areas of algebra. For example functional analysis and discrete mathematical programming have benefited from the theory. Another curious application of the theory is the abstract theory of measurement of physical quantities. The study of the Archimedean (f.o. or p.o.) property, homomorphisms, residuals (named as implication in this thesis), pseudocomplements and semilattice decompositions are of particular interest in f.o. and p.o. semigroups.

This dissertation consists of 4 chapters. In Chapter I, following the techniques and teminology of Saito, we extend some results of Saito and Satyanarayana on 0-Archimedean (Archimedean) f.o. semigroups to p.o. semigroups. The structure of naturally partially ordered semigroups (n.p.o.) are discussed and new characterizations for non-trivial n.p.o. semigroups are obtained. In Chapter II, following the ideas of Nemitz, we introduce the notion of negatively partially ordered implicative semigroups. Homomorphisms are studied between this class of semigroups. Some results of Nemitz on implicative semilattices are generalized and amplified to the case of implicative semigroups. The relationship between implicative homomorphisms and order filters is also investigated. In Chapter III, we study implicative abelian semigroups which are pseudo-complemented. Some of these ideas are due to Blyth. The relationship between a semigroup and its set of pseudocomplements is found. In Chapter IV, a specific class of semigroups, namely, the pseudo-indexed semigroups, is introduced. Following the idea of Petrich, we build this class of semigroups into a strong semilattice of semigroups. A construction theorem of pseudo-indexed semigroups is obtained and an isomorphism theorem on this class of semigroups is also established.

Chapter I

Partially Ordered Archimedean Semigroups

§0. Introduction

Let S be a partially ordered semigroup. Following the terminology of T. Saito [17], an element x of S is said to be non-negative (non-positive) if $x \leq x^2$ ($x^2 \leq x$) . If every element of S is non-negative (non-positive), then S is called non-negatively (non-positively) ordered semigroup. S is said to be positively ordered (negatively ordered) if $ab \geq a$ and $ab \geq b$ ($ab \leq a$ and $ab \leq b$) for every element a, b in S . In accordance with the concept of Archimedean classes defined by T. Saito, a partially ordered semigroup S is said to be O-Archimedean (that is, Order Archimedean) if, for every x and y in S , there exist natural numbers p, q such that $x^p \leq y^q$. A positive (negative) ordered semigroup S is called right (left) naturally partially ordered (abbreviated n.p.o.) if $a \lneq b$ implies that $b = ax$ ($b = ya$) for some elements x or $y \in S$. A right and left n.p.o. semigroup is called a n.p.o. semigroup, as defined by L. Fuchs in [7 ; 154]. There exists right n.p.o. semigroups which are not n.p.o. semigroups.

Totally ordered O-Archimedean and Archimedean semigroups and naturally totally ordered semigroups were studied by O. Hölder [11] and E.V. Hungtington [12] dated back to 1902 and 1904. In recent years, M. Satyanarayana ([20]; [21]) studied the class of complete totally ordered semigroups satisfying some weaker conditions other than the condition of naturally totally ordering. By modifying some work by T. Tamura [22] and T. Saito [18], he showed that under certain conditions, these semigroups are 0-Archimedean. In this chapter, we shall extend some of his results on totally ordered 0-Archimedean and Archimedean semigroups to partially ordered 0-Archimedean and Archimedean semigroups. The structure of n.p.o. semigroups are investigated and some new characterizations for non-trivial n.p.o. semigroups are obtained.

Throughout this chapter, S is always a non-trivial partially ordered semigroup. The non-trivity of S means $|S| \geq 2$. As usual, we write $b \geq a$ for $a \leq b$, and $a < b$ (or $b > a$) to mean that $a \leq b$ and $a \neq b$. If neither $a \leq b$ nor $b \leq a$, then a and b are called incomparable and is denoted by $a \| b$. On the other hand, if a, b are comparable, then we write $a \not\parallel b$. The neutral element, the zero, and the greatest element of S if they exist are denoted by the symbols e, o and u respectively. The reader is referred to L. Fuchs [7] for all other terminology and definitions not given here.

§1. 0-Archimedean P.O. semigroups

According to N.G. Alimov [1], two distinct elements a, b of an ordered semigroup S are said to form an anomalous pair if $a^n < b^{n+1}$; $b^n < a^{n+1}$ for all $n > 0$ or $a^n > b^{n+1}$; $b^n > a^{n+1}$ for all $n > 0$. Clearly, the first alternative may occur if $a, b \in P$, the positive cone of S and the second if $a, b \in N$, the negative cone of S (see [7]; p.162). It had been demonstrated by Hion [10] and Clifford [4] that the existence of anomalous pairs in an ordered semigroup provides useful information for the structure of the semigroup. Inspired by the definition of "anomalous pair", we give the following definition.

Definition 1.1 Let S be a p.o. semigroup. If a is a non-negative element of S and b is a non-positive element of S , then $\{a, b\}$ is called an Alimov pair.

Note. In the Alimov pair $\{a, b\}$, the elements a, b need not be distinct elements.

In this section, we shall show that the existence of Alimov pair in a p.o. semigroup also provides useful information for the structure of 0-Archimedean p.o. semigroup.

Lemma 1.2 Let S be an 0-Archimedean P.O. semigroup. Then the product of the elements of an Alimov pair is an idempotent, that is, if $\{x, y\}$ is an Alimov pair of S then $(xy)^2 = xy$.

Proof: By the 0-Archimedean property, there exists $p, q \in \mathbb{N}$ such that $x^p \leq y^q$. Since x is non-negative and y is non-positive, we have $x \leq x^p \leq y^q \leq y$, that is, $x \leq y$. However, by the isotone property of the p.o. semigroup S , $x \leq y$ implies $x^2 \leq xy$ and $yx \leq y^2$. As x is non-negative and y is non-positive, we thus have $xy \leq x^3y = x^2(xy) \leq (xy)(xy) = x(yx)y \leq xy^2y \leq xy$. Hence $xy = (xy)^2$.

Lemma 1.3 Let S be an 0-Archimedean p.o. semigroup such that $x \not\parallel x^2$ for all $x \in S$, then the following conditions are equivalent.

(1) S contains an idempotent f .

(2) The set of all non-negative elements of S is non-empty and has a greatest element g .

(3) The set of all non-positive elements of S is non-empty and has a smallest element h .

(4) S contains a zero element.

(5) S is not elementwise torsion-free, that is, the order of each element of S is finite.

(6) S contains an element with finite order.

(7) S contains an Alimov pair.

Note. This lemma points out an interesting fact that in an 0-Archimedean semigroup S with $x \# x^2$ for all $x \in S$, the locally finiteness property (6) implies the globally finiteness property (5).

Proof: (1) ⇒ (2) Let $f^2 = f \in S$. Then f is clearly a non-negative element of S. Let C be the set of all non-negative elements of S. $C \neq \emptyset$ since $f \in C$. As S is 0-Archimedean, for any $x \in C$, there exists $p, q \in \mathbb{N}$ such that $x \leq x^p \leq f^q = f$. This implies that f is the greatest element in C.

(2) ⇒ (1) Let g be the greatest element in C, the set of all non-negative elements in S. Since g is non-negative, $g \leq g^2$ which implies that $g^2 \leq g^3 = g(g^2) \leq g^4$. Thus g^2 is still non-negative and so $g^2 \in C$. However, by the maximality of g in C, $g^2 \leq g$. Hence $g = g^2$.

(1) ⇔ (3) Can be proved likewise as (1) ⇔ (2). Thus, we have proved (1) ⇔ (2) ⇔ (3).

(3) ⇒ (4) By assumption, $x \# x^2$ for all $x \in S$. Thus we have $x \leq x^2$ or $x \geq x^2$. If $x \leq x^2$ then, by (3) ⇔ (2), $x \leq f$. This fact implies that $fx \leq f$ and $xf \leq f$. Since $(xf)^2 = x(fx)f \leq xf^2f = xf$, by the uniqueness of f, we therefore have $xf = f$. Similarly, we can prove that $fx = f$. By using similar arguments, we can also show that $xf = fx = f$ for the case $x \geq x^2$. Thus (4) is proved.

(4) ⇒ (1) Trivial.

(4) ⇒ (5) Let f be the zero of S. Clearly $f^2 = f$. Since

S is 0-Archimedean, then there exists $p, q \in \mathbb{N}$ such that $x^p \leq f \leq x^q$ for all $x \in S$. In case if $p > q$, then $x^p \leq f = fx^{p-q} \leq x^q \cdot x^{p-q} = x^p$. In case if $q \geq p$, then $x^q = x^{q-p}x^p \leq x^{q-p}f = f \leq x^q$. Thus each of these cases shows that the order of x is finite.

(5) $\Rightarrow$ (6) Trivial.

(6) $\Rightarrow$ (7) Let $x \in S$ such that the order of x is finite. Since $x \nparallel x^2$, there exists $n \in \mathbb{N}$ such that $x^n = x^{n+1} = \ldots = x^{2n} = \ldots$. Write $x^n = f$. Then $f^2 = f$ and f is clearly both non-negative and non-positive. Thus $\{f, f\}$ is an Alimov pair of S.

(7) $\Rightarrow$ (1) Let $\{x, y\}$ be an Alimov pair of S. Then, by Lemma 1.2, $(xy)^2 = xy$.

The cycle of proof is completed.

Totally ordered 0-Archimedean semigroups with Alimov pairs were investigated by T. Saito [17]. Invoke a result of him, we are now able to extend a result of M. Satyanarayan in [21] on totally ordered semigroups to p.o. semigroups.

<u>Definition 1.4</u> A p.o. semigroup S is said to be weakly positive (weakly negative) if for all $x, y \in S$, either $x \| xy$ or $x \leq xy$, $x \lneqq x^2$ ($x \geq xy$, $x^2 \lneqq x$).

<u>Theorem 1.5</u> Let S be an 0-Archimedean p.o. semigroup such that $x \nparallel x^2$ for all $x \in S$. Then S is one of the following types:

(i) S is a nil semigroup.

(ii) S is a weakly positive p.o. semigroup.

(iii) S is a weakly negative p.o. semigroup.

Proof: If S contains idempotents, then by Lemma 1.3, we know that S has a zero element f and every element x of S is of finite order, that is, $x^m = f$ for some $m \in \mathbb{N}$. Hence S is nil. Now consider the case when S does not contain idempotents. Suppose if possible that S contains an Alimov pair $\{x, y\}$. Then by Lemma 1.2 $(xy)^2 = xy$, which contradicts S contains no idempotents. Invoke a result of Saito [18, Lemma 2.5], S is known to be either non-positively ordered (or non-negatively ordered) in strict sense i.e. $x \gneq x^2$ $(x \lneq x^2)$ for every x in S. Suppose that S is strictly non-positive and suppose $x \lneq xy$ for some $x, y \in S$. Then by the isotone property of S, we have $xy \leq xy^2$. Also, as $y^2 \leq y$, we have $xy^2 \leq xy$. Thus $xy^2 = xy$ and hence $xy^k = xy$ for all $k \in \mathbb{N}$. Because S is 0-Archimedean, there exists $p, q \in \mathbb{N}$ such that $x \geq x^p \geq y^q$. This implies that $x > x^2 \geq xy^q = xy$, a contradiction. Thus $x \| xy$ or $x \geq xy$ for all $x, y \in S$. This shows that S is a weakly negative p.o. semigroup. Adopting similar reasoning in the second case, wecan show that S is a weakly positive p.o. semigroup.

Remark 1. Theorem 1.5 is an extended result of M. Satyanaryana obtained in [21; Theorem 3] from Archimedean t.o. semigroups to p.o. semigroups.

Remark 2. It should be noted that the condition (ii) or (iii) alone does not imply the 0-Archimedean property of S. An example cited by M. Satyanarayan in [21] for t.o. semigroups is also a weakly positive p.o. semigroup but not 0-Archimedean.

§2. Archimedean P.O. semigroups

Recall that a p.o. semigroup S is said to be positively partially ordered (negatively partially ordered) if for all $a, b \in S$, $ab \geq a$ and $ab \geq b$ ($ab \leq a$ and $ab \leq b$). It is obvious to see that if S contains a neutral element e, then e must be the least element of S. We call a p.o. semigroup with neutral element e to be Archimedean if $a \leq b^n$ for some $n \in \mathbb{N}$, where a, b are non-neutral elements of S.

A right (left) naturally partially ordered semigroup is positively partially ordered and for all a, b in S with $a \lneq b$, there exists an element c of S such that $ac = b$ (or there exists $d \in S$ such that $da = b$). We shall abbreviate the class of naturally partially ordered semigroups by n.p.o. semigroups. The reader should aware that "n.p.o. semigroups" does not mean "negatively partially ordered" semigroups.

Naturally totally ordered semigroups which are 0-Archimedean had been completely characterized by O. Hölder and A.H. Clifford (see [4]). The theory of t.o. semigroups with a systematic treatment of the Archmiedean case was studied by L. Fuchs [7]. In this section, we shall extend some results of Fuchs [7] and Clifford [4] for n.p.o. semigroups which are Archimedean.

<u>Definition 2.1</u> A p.o. semigroup is said to be right (left) partial cancellative if for all $a, b \in S$ such that $ac = bc$ ($ca = cb$), then $a = b$ or $a \,/\!/\, b$.

<u>Definition 2.2</u> A p.o. semigroup S is said to be non-maximal right (left) partial cancellative, if S has a greatest element u and for all elements a, b, c in S, $ac = bc \neq u$ ($ca = cb \neq u$) then $a = b$ or $a \,/\!/\, b$.

Following the idea of A.H. Clifford ([4] or [7 ; p.164]), we have the following result for n.p.o. semigroups.

Lemma 2.3 Let S be an Archimedean right n.p.o. semigroup and S is not left partial cancellative. Then

(1) S contains a greatest element u .

(2) for every $a \in S$, there exists a $k \in \mathbb{N}$ such that $a^k = u$.

(3) S is non-maximal left (right) partial cancellative.

Proof: Essentially the proof is similar to A.H. Clifford's proof, but we have to make some minor changes.

(1) By hypothesis, three elements $a, b, c \in S$ exist such that $ca = cb$ and $a \lneq b$. Since $a \lneq b$ implies $b = at$ for some $t \neq e$. Hence $cb = c(at) = (ca)t = (cb)t$. Therefore $cb = (cb)t^n$ for all $n \in \mathbb{N}$. This implies that $cb = e$ or $cb = t^m = t^{m+1}$ for some $m \in \mathbb{N}$. As $cb = e$ implies $b = c = a = e$, which contradicts to $a \lneq b$. Hence we only have $cb = t^m = t^{m+1}$ for some $m \in \mathbb{N}$. By the Archimedean property, we have $cb \geq x$ for all $x \in S$. Thus cb must be the greatest element of S .

(2) This part follows at once from (1) and the Archimedean property of S .

(3) Suppose $cb = ca$ with $a \lneq b$. Then, as in the proof of (1), we have $cb = u$, the greatest element of S . However, this conclusion contradicts our hypothesis. Thus the p.o. semigroup satisfy the non-maximal left partial cancellative law.

Lemma 2.4 Let S be a non-trivial Archimedean right n.p.o. semigroup. Then S contains a greatest element u if and only if S is not left partial cancellative.

Proof: (Necessity) Suppose S contains a greatest element u. Since S is non-trivial, $|S| \geq 2$. Thus there exists an element a in S such that $a \lneqq u$. As S is positively ordered, $u \leq ua$ and $u \leq u \cdot u$. By the maximality of u, we have $ua = uu (= u)$ and $a \lneqq u$. This shows that S is not left partial cancellative.

(Sufficiency) See Lemma 2.3.

From the above two lemmas, we obtain the following characterizations for certain Archimedean right n.p.o. semigroups.

Theorem 2.5 (i) An Archimedean non-trivial right n.p.o. semigroup is non-maximal left partial cancellative if and only if it is a nil semigroup.

(ii) An Archimedean right n.p.o. semigroup S is left partial cancellative if and only if $\bigcap_{n=1}^{\infty} x^n S = \bigcap_{n=1}^{\infty} Sx^n = \emptyset$ for every element x in S.

Proof: (i) (Necessity) Suppose S is non-maximal left partial cancellative semigroup. then S contains a greatest element u for $|S| \geq 2$. By Lemma 2.4, S is therefore not left partial cancellative. This means that for every element x in S, $xu \geq u$ and $ux \geq x$, that is, $xu = u = ux$ by the maximality of u. Hence u is a zero element of S. by Lemma 2.3(2), S is a nil semigroup.

(Sufficiency). Suppose S is a non-trivial nil semigroup with 0 as the greatest element. Because S is positively ordered, 0 is obvious the greatest element of S. By Lemma 2.4, S does not satisfy the left partial cancellative law. Therefore, from Lemma 2.3(3), we know that S is a non-maximal left partial cancellative semigroup.

(ii) (Necessity) By Lemma 2.4, S does not contain a greatest element. Suppose if possible that $\bigcap_{n=1}^{\infty} x^n S \neq \emptyset$ for some $x \in S$. Let $t \in \bigcap_{n=1}^{\infty} x^n S$. Since S is positively ordered, $t \geq x^n$ for all $n \in \mathbb{N}$. However, by the Archimedean property of S, $t \leq x^m$ for some $m \in \mathbb{N}$, $t \lneqq x^m$ is clearly absurd. If $t = x^m$, then for every $n \geq m$, $x^n \leq t = x^m$. This implies $t = x^n$ for all $n \geq m$ and t is the greatest element of S. This contradicts S does not contain a greatest element. Thus $\bigcap_{n=1}^{\infty} x^n S = \emptyset$ for every element x in S. Similarly, we can prove that $\bigcap_{n=1}^{\infty} Sx^n = \emptyset$ for every element x in S.

(Sufficiency) Suppose $\bigcap_{n=1}^{\infty} Sx^n = \bigcap_{n=1}^{\infty} x^n S = \emptyset$ for every element x in S. Then by Lemma 2.4, S has a greatest element a for S is not left partial cancellative. This implies that $\bigcap_{n=1}^{\infty} Su^n = \{u\} \neq \emptyset$, which contradicts our hypothesis. thus, S is a left partial cancellative semigroup.

Remark It was noticed by M. Satyanarayan in [21; Lemma 2] that if S is a right n.t.o. semigroup. Then the condition $\bigcap_{n=1}^{\infty} x^n S = \bigcap_{n=1}^{\infty} Sx^n = \emptyset$ for every x in S is also a necessary and sufficent condition for S to be 0-Archimedean. Thus the condition $\bigcap_{n=1}^{\infty} x^n S = \bigcap_{n=1}^{\infty} Sx^n = \emptyset$ provides some useful information for the structure of naturally ordered semigroups.

Corollary 2.6 Every Archimedean right (left) n.p.o. semigroup without zero is strictly isotonic, that is, $a \lneqq b$ implies $ca \lneqq cb$ ($ac \lneqq bc$) for every element c in S.

Proof: Suppose there exists $t \in S$ such that $ta = tb$. Then by the naturally ordering property of S, $a \lneqq b$ implies $b = as$ for some $s \in S$. Thus $ta = tb = tas$. Similarly, $ta = ((ta)s)s = (ta)s^2$.

Continue this process, we have $(ta) = (ta)s^n$ for every $n \in \mathbb{N}$. That is, $s^n \leq (ta)$ for all $n \in \mathbb{N}$. However, by the Archimedean property of S, we have $ta \leq s^m$ for some $m \in \mathbb{N}$. Hence $ta = s^k = s^{k+1}$ for some $k \in \mathbb{N}$. Again, by the Archimedean property of S, we have $x \leq s^t$ for every $x \in S$ and some $t \in \mathbb{N}$. Thus we have shown that $x \leq ta$ and ta is the greatest element of S. Moreover, $ta \leq x(ta) \leq s^t s^k = s^{t+k} = ta$. Thus $x(ta) = ta$. Similarly $(ta)x = ta$. Hence ta is the zero element of S, which is absurd. Therefore $a \lneq b$ implies that $ca \lneq cb$ for every element c in S.

<u>Proposition 2.7</u> Let S be a non-trivial Archimedean right n.p.o. semigroup. Then $Sb \subset bS$ for every $b \in S$.

Proof: If $b = u$ or e then clearly $Sb \subset bS$. If $b \neq u$ or e, then for all a in S, we have $ab \geq b$. If $ab \gneq b$ then, by the right n.p.o. property, we have $ab = bt$ for some $t \in S$, thus $ab \in bS$. If $ab = b$, then $b = a^n b$ for all $n \in \mathbb{N}$. This implies that $b = u$ or e, which is absurd. therefore, $Sb \subset bS$ for all $b \in S$.

<u>Definition 2.8</u> A proper ideal I of a semigroup S is called completely semiprime if $x^n \in I$ implies $x \in I$, where $n \in \mathbb{N}$.

The following theorem is a characterization for n.p.o. semigroups to be Archimedean.

<u>Theorem 2.9</u> A right n.p.o. semigroup S with neutral element e and zero 0 is Archimedean if and only if $P = S\backslash\{e\}$ is the only completely semiprime ideal of S.

Proof: <u>(Necessity)</u> Let $P \subsetneq S$ be a completely semiprime ideal of the Archimedean p.o. semigroup S. Then for all $p \in P$ and for any $x \neq e$ in S, we have $x^n \geq p$ for some $n \in \mathbb{N}$. By the n.p.o. property of S, we have $x^n = p \in P$ or $x^n = pt \in P$ for some $t \in S$. Hence $x^n \in P$. Because P is completely semiprime, we have $x \in P$. Hence $S \backslash \{e\} \subset P$. Trivially, $P \subset S \backslash \{e\}$. Thus $P = S \backslash \{e\}$.

(Sufficiency) Let $P = S \backslash \{e\}$ be the only completely semiprime ideal of S. Consider the set $A = \{x \in S \mid \text{for every } y \in S,\ x^n \geq y \text{ for some } n \in \mathbb{N}\}$. Then $A \neq \emptyset$ as $0 \in A$. Pick any $x \in A$, $t, y \in S$. By the property that S is positively ordered and by the property of the set A, we have $tx \geq x$ and $x^n \geq y$ for some $x \in \mathbb{N}$. In addition, by the isotonic property of the p.o. semigroup S, we have $(tx)^n \geq x^n \geq y$. Thus, $tx \in A$ for any $t \in S$ and $x \in A$. This shows that A is a left ideal of S. Similarly, we can prove that A is a right ideal of S. Hence, A is a two-sided ideal of S.

We now show that A iscompletely semiprime. For this purpose, suppose $x^m \in A$ for some $x \in S$ and $m \in \mathbb{N}$. Then for all $y \in S$, we have $(x^m)^n \geq y$ for some $n \in \mathbb{N}$, by the definition of A. However, $(x^m)^n \geq y$ implies $x^{mn} \in A$ and therefore $x \in A$. Thus A is completely semiprime and so by assumption, $A = S \backslash \{e\}$. This shows that S is Archimedean.

§3. Characterization for right n.p.o. semigroups

Archimedean n.t.o. semigroups were completely characterized by Clifford and Hölder (see [11]). It was also noticed by Fuchs and Hölder [7 ; p.164] that their result can be proved with a weaker hypothesis, by replacing n.t.o. by right n.t.o. condition. Recently, without the full force of n.t.o. condition, M. Satyanarayana and C.S. Nagore [14] improved some results of Hungtington ([7 ; p.167]) for right n.t.o. semigroups without neutral element. In this section, we shall study Archimedean right n.p.o. semigroups and give an interesting charcterization for right n.p.o. semigroups.

Theorem 3.1 Let S be a non-trivial Archimedean right n.p.o. semigroup without neutral element. Then the following conditions are equivalent.

(1) S does not satisfy the left partial cancellative law.

(2) S contains a greatest element u .

(3) S has a zero element.

(4) S is nil.

(5) S satisfies the non-maximal left partial cancellative law.

(6) S contains at least an idempotent.

Proof: (1) $\Rightarrow$ (2): Suppose S is not left partial cancellative. Then there exists elements a, b and c in S such that $ca = cb$ and $a \lneq b$. By the right n.p.o. property, we have $b = at$ for some $t \in S$. Hence $ca = ca(t)$. As a consequence, we have $ca = [(ca)t]t = (ca)t^2$, and so $ca = (ca)t^n$ for every $n \in \mathbb{N}$. Write $u = ca$. Then $ut^n = u$ for all $n \in \mathbb{N}$. Hence $u \leq ux \leq ut^m = u$, that is, $u = ux$ and $u \geq x$. Thus u is the greatest element in S .

(2) $\Rightarrow$ (3): Let u be the greatest element of S . Then, because S is positively ordered, for all $u \in S$, we have $xu \geq u$ and $ux \geq u$. These imply that $xu = ux = u$, for u is the greatest element. Hence u is also the zero of S .

(3) $\Rightarrow$ (4): Since S is Archimedean, then for all $x \in S$,

there exists $n \in \mathbb{N}$ such that $x^n \geq u$. As the zero element u is also the greatest element of S, so $x^n = u$. Hence S is nil.

(4) $\Rightarrow$ (5): Referred to Theorem 2.5(i) (Note that Theorem 2.5 does not require S has the neutral element).

(5) $\Rightarrow$ (6): Clearly the greatest element u of S is an idempotent.

(6) $\Rightarrow$ (1): Let $f^2 = f \in S$. Then, as S is Archimedean, so for all $x \in S$, there exists $p, q \in \mathbb{N}$ such that $x \leq x^p \leq f^q = f$. This implies that f is the greatest element of S.

By (2) $\Rightarrow$ (3), observe that f is the zero element of S. Thus we have $f = f^2 = fa$ for all $a \lneq f$. Hence S does not satisfy the left partial cancellative law.

Parallel to the above theorem, we have the following result which exhausts all the other possible cases.

<u>Theorem 3.2</u> Let S be an Archimedean right n.p.o. semigroup without neutral element. Then the following conditions are equivalent.

(1) S satisfies the left partial cancellative law.

(2) for all $x \in S$, $\bigcap_{n=1}^{\infty} x^n S = \bigcap_{x=1}^{\infty} Sx^n = \emptyset$.

(3) S contains no idempotents at all.

Proof: (1) $\Leftrightarrow$ (2) Referred to theorem 2.5(ii).

(1) $\Leftrightarrow$ (3) Referred to the above Theorem 3.1 ((1) $\Leftrightarrow$ (7)).

We now give a characterization for right n.p.o. semigroups. We begin with some definitions.

Definition 3.3 An ideal I of a p.o. semigroup is called convex if $a, b \in I$, then $c \in I$ for all c satisfying $a \leq c \leq b$.

Definition 3.4 Let A, B be subsets of a partially ordered set S such that $A \nsubseteq B$ and $B \nsubseteq A$. Then $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is said to be discrete if for all $x \in A \backslash B$, $y \in B \backslash A$, $x \# y$.

Lemma 3.5 Let S be a right n.p.o. semigroup. Then the following statements are true.

(1) All right ideals of S are convex;

(2) Right ideals of S are two-sided;

(3) If S contains a neutral element e, then the only element x such that $xy = e$ is e itself;

(4) x is a greatest element of S if and only if x is a zero of S;

(5) $S \backslash S^2 = \{x \in S \mid x$ is a non-idempotent minimal element of $S\}$.

(6) If A and B are right ideals of S such that $A \nsubseteq B$, $B \nsubseteq A$, then $A \triangle B$ is discrete;

(7) If all minimal elements of S are idempotents then S is globally idempotent, that is, $S^2 = S$.

Proof: Referred to M. Satyanarayana and C.S. Nagore [14; Proposition 1] with minor changes if necessary.

Theorem 3.6 Let S be a positively p.o. semigroup. then the following statements are equivalent:

(1) S is a right n.p.o. semigroup.

(2) Let a, b be elements of S such that $aS^1 \nsubseteq bS^1$ and $bS^1 \nsubseteq aS^1$, then $aS^1 \triangle bS^1$ is discrete. (where S^1 is the semigroup adjoined with the neutral element e).

(3) If A and B are right ideals of S such that $A \nsubseteq B$, $B \nsubseteq A$ then $A \triangle B$ is discrete.

(4) All right ideals of S are convex.

Proof: (1) $\Rightarrow$ (2) Suppose $aS^1 \nsubseteq bS^1$ and $bS^1 \nsubseteq aS^1$. Then we can pick $x \in aS^1 \backslash bS^1$ and $y \in bS^1 \backslash aS^1$. Suppose $x \lneq y$. Then, by the natural ordering, there exists $t \in S$ such that $y = xt$. This implies $y \in xS^1 \subset aS^1$ which is a contradiction. Similarly, we can prove that $y \nleq x$. Therefore $x \mathbin{/\!/} y$ and $aS^1 \triangle bS^1$ is hence discrete.

(2) $\Rightarrow$ (3) Suppose $A \nsubseteq B$ and $B \nsubseteq A$. Let $x \in A \backslash B$ and $y \in B \backslash A$ respectively. Then we have $xS^1 \nsubseteq yS^1$ and $yS^1 \nsubseteq xS^1$. By (2), x and y is incomparable, that is, $x \mathbin{/\!/} y$. Since x and y are arbitrarily taken from $A \backslash B$ and $B \backslash A$ respectively, $A \triangle B$ is therefore discrete.

(3) $\Rightarrow$ (1) Suppose $x \lneq y$ for $x, y \in S$. It is clear to see that $x \notin yS^1$ as S is positively prdered. Suppose $y \notin xS^1$. Then by (3), $xS^1 \triangle yS^1$ is in particular, discrete, that is, $x \mathbin{/\!/} y$, which is a contradiction. Hence $y \in xS^1$ and so $y = xt$ for some $t \in S$. This shows that S is a right n.p.o. semigroup.

(1) $\Rightarrow$ (4) Referred to Lemma 3.5.

(4) $\Rightarrow$ (3) Let A and B be right ideals of S such that $A \nsubseteq B$, $B \nsubseteq A$. Take $x \in A \backslash B$ and $y \in B \backslash A$. Suppose if possible that $x \lneq y$, then $x \lneq y \leq xy$. Because $x \in A$ and $xy \in A$, we have $y \in A$ by the convexity of A. This is clearly a contradiction. Hence $x \nleq y$. Similarly, we can show that $y \nleq x$. Thus $x \mathbin{/\!/} y$ and $A \triangle B$ is discrete.

This completes the cycle of proofs.

Remark. Right n.t.o. semigroups have been studied in detail by M. Satyanarayana and C.S. Nagore in [14]. They pointed out that right ideals in right n.t.o. semigroups are convex and they are linearly ordered under set-inclusion. As a converse, they showed that, under the left cancellative law, there are certain classes of monoids satisfying the linearly ordering of right ideals can be endowed with a n.t.o. structure. Of course, if the semigroup S is not totally ordered, the right ideals of S need not form a chain under set-inclusion. Suprisingly, in our theorem 3.6, we show that a positively ordered p.o. semigroup S is right n.p.o. if and only if every right ideal of S is convex. Thus the convexity of the right ideals is more important than the cancellative property in p.o. semigroups.

§4. Noetherian Archimedean n.t.o. and n.p.o. semigroups

A semigroup S is said to be right Noetherian if and only if every right ideal of S is finitely generated. Not all n.t.o. semigroups are Noetherian, for example, the set S of all positive real numbers under addition and with the usual ordering is a Non-Noetherian n.t.o. semigroup. However, there are examples of Noetherian n.t.o. semigroups. For example, the infinite cycle semigroup generated by a singleton element x with $x^n < x^{n+1}$ for every $x \in \mathbb{N}$ is a right Notherian n.t.o. semigroup. This class of p.o. semigroups were studied by M. Satyanarayana and N.S. Nagore in [14]. The following characterization theorem was proved by them. We include here their theorem for the sake of completeness.

<u>Theorem 4.1</u> [14] Let S be a right cancellative right n.t.o. semigroup. Then the following are equivalent:

(1) S is an infinite cyclic semigroup.

(2) $S \neq S^2$ and S is a left Noetherian semigroup.

(3) $S \neq S^2$ and between any two elements, there are at most a finite number of elements.

(4) $\bigcap_{n=1}^{\infty} S^n = \emptyset$.

Proof: Clearly (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4).

(2) $\Rightarrow$ (1): As noted by Satyanarayana and Nagove in [14], $S \neq S^2$ and S is left Notherian. These imply that S has no idempotent and $S = x \cup xS$, where x is a minimal element of S . By the right cancellative condition, S does not contain any idempotent. Let $\langle x \rangle = \{x^n : n \geq 1\}$. If $y \notin \langle x \rangle$, then $y = xs_1$ for some $s_1 \notin \langle x \rangle$. Hence there exists $s_2 \notin \langle x \rangle \cup s_1$ such that $s_1 = xs_2$. Similarly, $s_2 = xs_3$ for some $s_3 \notin \langle x \rangle \cup \{s_1, s_2\}$. Inductively, there exists a sequence $\{s_n\}$ such that $s_n = xs_{n+1}$ and $s_n \gneq s_{n+1}$ for all n . There is also a chain of left ideals $S^1s_1 \quad S^1s_2 \subset \dots \subset S^1s_n \subset \dots$. By the left Noetherian condition, $S^1s_n = S^1s_{n+1}$ for some natural number n . Hence $s_{n+1} = ts_n$ for some $t \in S$ and $s_{n+1} = txs_{n+1} = (tx)^2 s_{n+1}$, which implies $tx = (tx)^2$ by the right cancellative condition. This contradicts S has no idempotent. Thus every element of S is a power of x .

(3) $\Rightarrow$ (1): In (2) $\Rightarrow$ (1), if $y \neq x^n$ for any n , then $s_1 \gneq s_2 \gneq \dots \gneq x$. This is absurd by hypothesis. Hence the conclusion is evident.

(4) $\Rightarrow$ (1): Since $\bigcap_{n=1}^{\infty} S^n = \emptyset$, $S \neq S^2$. As before, we have

$S = x \cup xS$ where $S \backslash S^2 = \{x\}$. Hence $S^2 = xS$ and inducitively $S^r = x^{r-1}S$ for all $r \geq 2$. Let $y \in S$, then there exists a natural number r such that $y \in S^r \backslash S^{r+1}$. Therefore, $y = x^{r-1}s$ where $s = x$ or $s = xt$. If $s = xt$, then $y \in x^r S = S^{r+1}$. Hence $y = x^r$.

Remark. In general, Theorem 4.1 does not hold for n.p.o. semigroups with right cancellative condition because the minimal elements of n.p.o. semigroups may not be comparable. For example, let $\{x_1, \ldots, x_n\}$ be a minimal generating set of a n.p.o. semigroup S , but the chain $x_1 < x_2 < \ldots < x_n$ does not exist.

We now note that there are examples of Noetherian n.p.o. semigroups which are not n.t.o. semigroups. (Of course, Noetherian n.t.o. semigroups are in particular Noetherian n.p.o. semigroup, but the converse is not true). It is easy to observe that every finite v-semilattice is Noetherian n.p.o. semigroup. The following is another example:

Example 4.2 Let $S = \{x^n, y^n : n \geq 1\}$ with a relation $\leq$ such that (i) $x^i \leq x^j$ if and only if $i \leq j$; (ii) $x^i \gneq y^j$ iff $i \gneq j$ and (iii) $y^i \leq y^j$ if and only if $i \leq j$. Clearly $(S, \leq)$ is a partially ordered set. For all a and b in S, define

$$ab = \begin{cases} x^{i+j} & a = x^i, b = x^j \\ y^{i+j} & a = y^i, b = y^j \\ x^{i+j} & a = x^i, b = y^j \\ x^{i+j} & a = y^i, b = x^j \end{cases} .$$

Then $(S, \leq)$ is an abelian p.o. semigroup. For all a and b in S with $a \lneq b$, there exists $c \in S$ such that $b = ac$ and with the following properties: (i) if $a = x^i, b = x^j$, then $b = a \cdot x^{j-i}$; (ii) if $a = y^i, b = x^j$, then $b = a \cdot x^{j-i}$ and (iii) if $a = y^i$, $b = y^j$, then $b = a \cdot y^{j-i}$. Moreover, $a \leq ab$, $b \leq ab$ for all a and b in S . Hence, $(S, \leq)$ is a n.p.o. semigroup. For each ideal

A of S , let $p = \min\{k \geq 1 : x^k \in A\}$, $q = \min\{t \geq 1, y^t \in A\}$. Then $A = \{x^m, y^n : m \geq p, n \leq q\} = \langle\{x^m, y^n\}\rangle$, A is finitely generated. Since there are only finite numbers k and t such that $1 \leq k < p$ and $1 \leq t < q$, there are only finite numbers of ideals B such that $A \subsetneqq B$. So the chain of left ideals $A_1 \subsetneqq A_2 \subsetneqq \ldots \subsetneqq A_n \subsetneqq \ldots$ will be terminated for some n . Thus z is left Noetherian n.p.o. But $x^i \| y^i$ for every $i \geq 1$. Hence S is not n.t.o.

Recall that an element t of a partially ordered semigroup S is said to be the smallest element of S if for every s in S , $s \geq t$. The following theorem is due to M. Satyanarayana and C.S. Nagare [14].

<u>Theorem 4.3</u> Let S be a right n.p.o. semigroup without neutral element but containing the smallest element x . Then S is archimedean if and only if S is an infinite or a finite semigroup generated by x .

We now turn to study the left Noetherian archimedean n.p.o. semigroups. In particular, we show that every element of such a semigroup can be expressed as a product of minimal elements.

<u>Theorem 4.4</u> Let S be an archimedean n.p.o. semigroup without neutral element. If S is left Noetherian, then every element of S can be expressed as a product of minimal elements.

Proof: We first claim that for every non-minimal element $x \in S$, there exists a minimal element $m \lneqq x$. Suppose that this is not true. Then, there exists a non-minimal element s_1 in S such that $s_1 \lneqq x$. Denote x by s_o and repeat the same arguments, we find that there exists a sequence of elements $\{s_n\}$ such that $\ldots \lneqq s_n \lneqq \cdots \lneqq s_1 \lneqq x = s_o$. By the natural ordering property of S , we have $s_i = x_{i+1}s_{i+1}$ for some x_{i+1} in S and $i = 0, 1, \ldots, n, \ldots$. This implies that there exists an infinite sequence of left ideals of S such that $S^1s_0 \subsetneq S^1s_1 \subsetneq S^1s_2 \subsetneq \dots \subsetneq S^1s_n \subsetneq \dots$, which is a contradiction. Thus, for any non-minimal element x in S , we are able to find a minimal element m_1 in S such that $m_1 \lneq x$. Since S is naturally ordered, there exists s_1 such that $m_1 s_1 = s_0$. If s_1 is not a minimal element of S , then by the same argument, there exists a minimal element $m_2 \lneq s_1$ and an element s_2 such that $m_2 s_2 = s_1$. Since S is left Noetherian, we therefore obtain a sequence of finite elements $\{s_i : i = 1, 2, \dots, n\}$ such that $s_i = m_i s_{i+1}$ for $i = 1, 2, \dots, n-1$. As s_n is a minimal element of S , so $x = m_1 m_2 \dots m_{n-1} s_n$, where $m_1, m_2, \dots, s_n$ are minimal elements of S . Thus the proof is completed.

Corollary 4.5. A left Noetherian archimedean n.p.o. semigroup S without neutral element is generated by a set of minimal elements of S .

Corollary 4.6. If the set of minimal elements of the left Archimedean n.p.o. semigroup S under multiplication are commutative, then S is a commutative semigroup.

Corollary 4.7. If S has a zero element then $S = \langle S \setminus S^2 \rangle$.

Proof: Since S is archimedean n.p.o. semigroup without identity, so S has at most one idempotent. Thus, by Theorem 4.1, $S \setminus S^2 = \{x \in S : x \text{ is a minimal element}\}$. Hence $S = \langle S \setminus S^2 \rangle$.

Chapter II

Homomorphisms of Implicative Semigroups

§0. Introduction

An implicative semi-lattice is an algebraic system having as models logical systems equipped with implication and conjunction, but not possessing a disjunction. The position of implicative semi-lattices in algebraic logics was clearly displayed by H.B. Carry in [5] and the relation of implicative lattices to Brouwerian logics was explained by Garrett Birkhoff [2]. Implicative semilattices were systematically studied by W.C. Nemitz [15]. In his paper, he showed that certain results for Brouwerian logics (equipped with disjunction) obtained by V. Glivenko can be proved for implicative semilattices. Also the relationship between homomorphisms of implicative semilattices and their kernels were investigated by him. In [3], T.S. Blyth has generalized some results of Nemitz [15] by introducing the notion of a Brouwerian semigroups. The results of Blyth have been further generalized by M.F. Janowitz and C.S. Johnson Jr. [13].

In this chapter, following the ideas of Nemitz and Blyth, we introduce the notion of negatively partially ordered implicative semigroups and study the homomorphisms between these semigroups. Some results of Nemitz on implicative semilattices are generalized and amplified on implicative semigroups. The reader is referred to W.C. Nemitz [15] and G. Birkhoff [2] for all terminology and definitions not mentioned in this chapter.

§1. Implicative semigroups

By a negatively partially ordered semigroup is meant a set on which there is defined a partial ordering $\leq$ and an associative multiplication which is such that for all $x, y, z \in S$.

(i) $(xy)z = x(yz)$.

(ii) $x \leq y \Rightarrow (\forall z \in S) \; xz \leq yz$ and $zx \leq zy$.

(iii) $xy \leq x$ and $xy \leq y$.

Definition 1.1 A negatively partially ordered semigroup endowed with a binary operation $* : S \times S \longrightarrow S$ such that for any elements x, y, z of S, $z \leq x * y$ if and only if $zx \leq y$ is called an implicative negatively partially ordered semigroup. The operation $*$ is called implication. We shall usually refer to S as an implicative n.p.o. semigroup omitting reference to the negatively partially ordering and compositions.

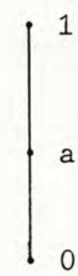
It should be noticed that if $(S, \cdot, \leq, *)$ is an implicative n.p.o. semigroup, then for every x of S, $x^2 \leq x$ implies that $x \leq x * x$. Moreover, for all $x, y \in S$, $(x * x)y \leq y$ implies that $x * x \leq y * y$ and by symmetry, $y * y \leq x * x$. Thus, an implicative n.p.o. semigroup S always contains a greatest element, namely, $x * x$.

Let 1 be the greatest element of a p.o. semigroup S if exists. If 1 is also the multiplicative identity of the n.p.o. semigroup S then by Satayanarayana and C.S. Nagore [14], we know that $xy = 1$ if and only if $x = y = 1$ for all $x, y \in S$. Therefore, we can always adjoin 1 to a n.p.o. semigroup S such that 1 is both the greatest element and the multiplicative identity of S .

The following example shows that the greatest element of a n.t.o. semigroup need not be the multiplicative identity of S even if S is implicative.

Example 1.2 Let $S = \{1, a, 0\}$ with Cayley table and Hasse diagram as follows:

$\cdot$	1	a	0
1	1	0	0
a	0	0	0
0	0	0	0

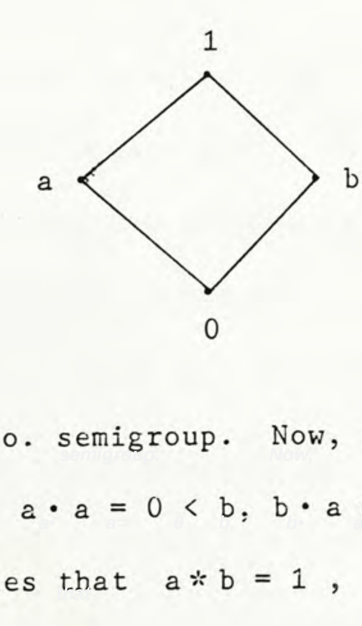


As $x * x = 1$ for all $x, y \in S$. Hence it is easy to see that S is an implicative n.t.o. semigroup. However, $1a = 0 \neq a$, so the greatest element 1 is not the multiplicative identity of S.

The following example points out that not every n.p.o. semigroup with 1 as its multiplicative identity admits the implicative structure.

Example 1.3 Let S be the set $\{0, 1, a, b\}$ with Cayley table and Hasse diagram as follows:

$\cdot$	1	a	b	0
1	1	a	b	0
a	a	0	0	0
b	b	0	b	0
0	0	0	0	0



Then it is easy to see that S is a n.p.o. semigroup. Now, let us consider $a * b$ for all $a, b \in S$. Clearly $a \cdot a = 0 < b$, $b \cdot a = 0 < b$. Therefore $a \leq a * b$ and $b \leq a * b$. This implies that $a * b = 1$, the greatest element of S . However, $1a = a \leq b$. This means that $a*b$ does not exist in S and S is not implicative.

In the rest of this chapter, we shall assume S is an implicative n.p.o. semigroup with its greatest element 1 as its multiplicative identity.

The following theorem extends the fundamental properties of implicative semilattices obtained by W.C. Nemitz [15] to implicative n.p.o. semigroups.

Theorem 1.4 Let S be an implicative n.p.o. semigroup. The following results are true for any elements x, y, z of S :

(1) $x \leq 1$, $x*x = 1$, $x = 1*x$.

(2) $x \leq x*x^2$.

(3) $x \leq y*x$.

(4) If $x \leq y$, then (a) $x*z \geq y*z$; (b) $z*x \leq z*y$.

(5) $x \leq y$ if and only if $x*y = 1$.

(6) $x*(y*z) = (xy)*z$.

(7) $x \leq y*(xy)$.

(8) if S is abelian, then $x*y \leq sx*sy$ for all s in S .

Proof: Essentially all these results were proved to be true in implicative semilattices by H.B. Curry [5] and W.C. Nemitz [15]. This theorem says that the corresponding results also true for implicative n.p.o. semigroups. For the convenience of the reader, we provide the proofs in detail.

Proof: (1) The results follow directly from the definition of implicative n.p.o. semigroup.

(2) Trivial as $x^2 \leq x^2$.

(3) Trivial as $xy \leq x$.

(4) (a) Obviously, $y*z \leq y*z$ implies $(y*z)y \leq z$. As $x \leq y$, so $(y*z)x \leq z$. Hence we have $y*z \leq x*z$.

(b) As $z*x \leq z*x$ implies that $(z*x)z \leq x \leq y$. Hence $z*x \leq z*y$.

(5) Suppose that $x \leq y$. Then $1 = x*x \leq x*y$. This implies that $x*y = 1$. Conversely, suppose that $x*y = 1$. Then $1 \leq x*y$ and so $x = 1 \cdot x \leq y$. Thus (5) is proved.

(6) Let $u = x*(y*z)$, $t = (xy)*z$. Then $ux \leq y*z$ implies that $u(xy) = (ux)y \leq z$. Hence we have $u \leq (xy)*z = t$. On the other hand, $(tx)y = t(xy) \leq z$ implies that $(tx) \leq y*z$. Hence $t \leq x*(y*z) = u$. Therefore we have shown that $t = u$.

(7) By (6), we have $x*[y*(xy)] = xy*(xy) = 1$, and by (5), we have $x \leq y*xy$. Thus (7) is proved.

(8) Let $t = (x*y)*(sx*sy)$ and $u = x*y$. Then by (6), we have $t = (x*y)sx*sy$. Also, $ux \leq y$ implies that $usx \leq sy$. Then apply (4)(a), we have $t = usx*sy \geq sy*sy = 1$, that is, $t = (x*y)*(sx*sy) = 1$. Thus by (5), we have $x*y \leq sx*sy$ for all $s \in S$.

As a result of Theorem 1.4, we can see that all results obtained by H.B. Curry [5] and W.C. Nemitz [15] are corollary of Theorem 1.4.

<u>Corollary 1.5</u> Let L be an implicative semilattice. Then the following results are true for any elements x, y, z of L :

(1) $x \leq 1$, $x*x = 1$, $x = 1*x$

(2) $x \leq y$ if and only if $x*y = 1$

(3) $y \leq x*y$

(4) if $x \leq y$, then $x*z \geq y*z$, and $z*x \leq z*y$

(5) $x * (y * z) = (x \wedge y) * z$

(6) $x * (y \wedge z) = x * y \wedge x * z$

(7) $x * (y * z) = (x * y) * (x * z)$

(8) if L is a lattice with least upper bound $\vee$; then L is distributive, and $(x \vee y) * z) = (x * z) \wedge (y * z)$.

§2. Homomorphism between implicative n.p.o. semigroups

Let $(S, \cdot, \leq, *)$ and $(S', o, \leq, *)$ be two implicative n.p.o. semigroups let α be a mapping from $(S, o, \leq, *)$ onto $(S', o, \leq, *)$ such that $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all elements x and y of S . Then α is called an implicative homomorphism of S onto S' .

Implicative homomorphisms between implicative meet semilattices have been thoroughly studied by W.C. Nemitz in [15]. In this section, we shall amplify his results to implicative n.p.o. semigroups.

In order to extend the result of Nemitz on implicative semilattices to implicative n.p.o. semigroups, we need to introduce the definition of order filters in n.p.o. semigroups.

Definition 2.1 A non-empty proper subset J of a n.p.o. semigroup S is called an order filter of S if and only if

(i) $xy \in J$ if and only if $x \in J$ and $y \in J$, where x, y are arbitrary elements of S .

(ii) if $x \in J$ and $y \geq x$, then $y \in J$.

[Note: By a filter J in a meet semilattice S , we mean J satisfies condition (ii) and part of condition (i), that is, we only require that "$x \wedge y \in J$ for any $x \in J$ and $y \in J$" . Thus, in this sense, the class of order filters in a meet semilattice is a subclass of filters.]

<u>Theorem 2.2</u> Let $(S, \cdot, \leq, *)$ and $(S', o, \leq, *)$ be two implicative n.p.o. semigroups. Let α be an implicative homomorphism from S onto S' . Then the following properties hold

(1) $\alpha(1) = 1'$, with $1'$ is the greatest element of S' as well as the multiplicative identity.

(2) α is isotonic, that is, if $x \leq y$ then $\alpha(x) \leq \alpha(y)$.

(3) α is a semigroup homomorphism, that is, $\alpha(x, y) = \alpha(x) \circ \alpha(y)$.

(4) $J = \alpha^{-1}(1')$ is an order filter of S .

(5) α is a semigroup isomorphism if and only if $J = \{1\}$.

Proof: (1) Apply Theorem 1.4(1), it is easy to verify that $\alpha(1) = \alpha(1 * 1) = \alpha(1) * \alpha(1) = 1'$.

(2) Suppose that $x \leq y$. Then as α is an implicative homomorphism, we have $\alpha(x) * \alpha(y) = \alpha(x * y) = \alpha(1) = 1'$. By Theorem 1.4(5), we therefore have $\alpha(x) \leq \alpha(y)$.

(3) We first show that $\alpha(xy) \leq \alpha(x) \circ \alpha(y)$. Since α is an onto mapping between S and S' , so there exists z in S such that α is an onto mapping between S and S' , so there exists z in S such that $\alpha(z) = \alpha(x) \circ \alpha(y)$. Now $\alpha(xy) * \alpha(z) = \alpha(xy * z) = \alpha[x * (y * z)] = \alpha(x) * \alpha(y * z) = \alpha(x) * [\alpha(y) * \alpha(z)] = [\alpha(x) \circ \alpha(y)] * \alpha(z) = \alpha(z) * \alpha(z) = 1$. This shows that $\alpha(xy) \leq \alpha(z) = \alpha(x) \circ \alpha(y)$.

Next, we show that $\alpha(x) \circ \alpha(y) \leq \alpha(x \cdot y)$. As $\alpha(y) * \alpha(xy) = \alpha(y * (xy))$, so by Theorem 1.4(7), we have $x \leq y * (xy)$. This implies that $\alpha(x) \leq \alpha(y * xy)$, that is, $\alpha(x) \leq \alpha(y) > \alpha(xy)$. Thus, $\alpha(x) \circ \alpha(y) \leq \alpha(xy)$.

Hence we conclude that $\alpha(xy) = \alpha(x) \circ \alpha(y)$. This proves (3).

(4) Suppose that $x \in J$ and $y \in J$. Then we have $\alpha(xy) = \alpha(x) \circ \alpha(y) = 1' \circ 1' = 1'$ and so $xy \in J$: If $xy \in J$. Then we have $x \geq xy$. This implies that $\alpha(x) \geq \alpha(x \cdot y) = 1'$ and $\alpha(x) = 1'$. Hence $x \in J$. Similarly, we can show that $y \in J$. Moreover, as $x \in J$, $x \leq y$. By the isotonic property of α , we have $\alpha(y) \geq \alpha(x) = 1'$. This implies $y \in J$. Thus J must be an ordered filter of S .

(5) Suppose that $J = \{1\}$ and $\alpha(x) = \alpha(y)$. Then we have $\alpha(x * y) = \alpha(x) * \alpha(y) = \alpha(x) * \alpha(x) = 1'$. This means that $x * y \in J$, that is, $x * y = 1$ and $x \leq y$. Apply samilar arguments, we can show that $y * x = 1$ and $y \leq x$. Hence $x = y$. In other words, α is a semigroup isomorphism between S and S' . The proof is thus completed.

<u>Theorem 2.3</u> Let L and L' be two implicative $\wedge$-semilattices. Let α be a mapping from L <u>into</u> L' , such that for all x and y elements of L , $\alpha(x * y) = \alpha(x) * \alpha(y)$. Let $J = \alpha^{-1}(1')$. Then the following properties hold:

(1) $\alpha(1) = 1'$

(2) α is an isotonic homomorphism

(3) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$ forany elements x and y of L .

(4) J is a (semilattice) filter of L .

(5) α is a 1-1 homomorphism if and only if the filter J of L is of the form $J = \{1\}$.

(6) If L and L' are both lattices, and if α is an <u>onto</u> mapping, then denoting least upper bounds by $\vee$, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, for any elements x and y of L .

Proof: (1) $\alpha(1) = \alpha(1*1) = \alpha(1)*\alpha(1) = 1'$.

(2) If $x \leq y$, then $\alpha(x)*\alpha(y) = \alpha(x*y) = \alpha(1) = 1'$. So $\alpha(x) \leq \alpha(y)$.

(3) Clearly $x \wedge y \leq x$ and $x \wedge y \leq y$ imply that $\alpha(x \wedge y) \leq \alpha(x) \wedge \alpha(y)$. On the other hand, we have $x*[y*(x \wedge y)] = (x \wedge y)*(x \wedge y) = 1$. This implies that $\alpha(x)*\alpha(y*(x \wedge y)) = 1'$ and so $\alpha(x) \leq \alpha(y*(x \wedge y)) = \alpha(y)*\alpha(x \wedge y)$. Therefore, $\alpha(x) \wedge \alpha(y) \leq \alpha(x \wedge y)$.

(4) and (5) follow from Theorem 2.2. For (6), see Nemitz [15].

Remark. The proof of (3) is essentially different from W.C. Nemitz. In particular, we prove (3) without assuming that α is surjective, but in carrying out his proof, W.C. Nemitz had to assume α to be surjective. Thus the assumption "α is surjective" is indeed superflous.

§3. The quotient structures of Implicative n.p.o. semigroups

In this section, we shall be concerned with implicative homomorphisms from Implicative abelian n.p.o. semigroups onto another implicative abelian n.p.o. semigroups which satisfy the conditions of Theorem 2.3, and have order filters for kernels. The construction of a quotient implicative n.p.o. semigroup will be discussed.

Let $(S, \cdot, \leq, *)$ be an implicative abelian n.p.o. semigroup and let J be an order filter of S . It is clear that $1 \in J$. Now, we define a relation "$\sim$" on S by saying that $x \sim y$ if and only if there exists an element $c \in J$ such that $cx \leq y$, $cy \leq x$ for all $x, y \in S$.

Lemma 3.1 The relation "$\sim$" defined above for implicative abelian n.p.o. semigroups S is a convex congruence relation defined on S.

Proof: It is trivial to see that $x \sim x$. Thus "$\sim$" is reflexive. Also, it can be easily verified that $x \sim y$ if and only if $y \sim x$. So, "$\sim$" is symmetric. For the transitivity, we assume that $x \sim y$ and $y \sim z$. Then by definition, there exists an element $c \in J$ such that $cx \leq y$, $cy \leq x$. Similarly, there exists an element $d \in J$ such that $dz \leq y$, $dy \leq x$. As S is an abelian n.p.o. semigroup, we therefore have $(cd)x = d(cx) \leq dy \leq z$, $(cd)z = c(dz) \leq cy \leq x$. As $cd \in J$, so by definition, we have $x \sim z$. The transitive law is proved. Hence "$\sim$" is an equivalence relation defined on S. Now, we show that "$\sim$" is also a congruence relation. Suppose that $x \sim z$. Then there exists c in J such that $cx \leq z$, $cz \leq x$. Hence $c(xs) = (cx)s \leq zs$, $c(zs) = (cz)s \leq xs$ for all $s \in S$. These imply that $zs \sim xs$ and $sz \sim sx$. To show that "$\sim$" is convex, we assume $x \leq y \leq z$ and $x \sim z$. Then there exists c in J such that $cx \leq z$, $cz \leq x$. Since $y \leq z$, so we have $cy \leq cz \leq x$. As $x \leq y$, clearly $cx \leq y$. this means that $x \sim y$. Thus "$\sim$" is a convex congruence relation of S.

In the rest of this section, we shall call "$\sim$" be the convex congruence relation defined on S induced by the order filter J. The quotient $S/\sim$ is clearly a semigroup under multiplication.

<u>Lemma 3.2</u> Let $(S, \cdot, \leq, *)$ be an implicative n.p.o. abelian semigroup. Then $(S/\sim, \cdot, (\leq), *)$ is also an implicative n.p.o. abelian semigroup.

Proof: We first define "$(\leq)$" on $S/\sim$ by $x/\sim (\leq) y/\sim$ if and only if for any $a \in x/\sim$, $b \in y/\sim$, there exists c in J such that

$ca \leq b$ in S, where $x/\sim$ and $y/\sim$ are the equivalence classes of x and y under "$\sim$" respectively. We now show that $(\leq)$ is a partial ordering. Clearly $x/\sim (\leq) x/\sim$ by the definition of "$\sim$" and $(\leq)$. Let $x/\sim (\leq) y/\sim$ and $y/\sim (\leq) x/\sim$. Then for any $h \in x/\sim$, $k \in y/\sim$, there exists elements c, d in J such that $ch \leq k$ and $dk \leq h$. Therefore we have $(cd)h = d(ch) \leq dk \leq k$, $(cd)k = c(dk) \leq ch \leq h$. Because $cd \in J$ and $h \sim k$, so $x/\sim$ is identically the same as $y/\sim$, that is $x/\sim (=) y/\sim$. If $x/\sim (\leq) y/\sim$, $y/\sim (\leq) z/\sim$, then for any $p \in x/\sim$, $q \in y/\sim$ and $r \in z/\sim$, we can find elements c, d in J such that $cp \leq q$, $dq \leq r$. These imply that $(cd)p = d(cp) \leq dq \leq r$. As $cd \in J$, so $x/\sim (\leq) z/\sim$. Thus $(\leq)$ is indeed a partial ordering defined on $S/\sim$. Now let us define the multiplication $S/\sim$ by $x/\sim \cdot y/\sim (=) xy/\sim$. Obviously, this is a semigroup multiplication. Moreover, $x/\sim \cdot y/\sim (\leq) x/\sim$ and $y/\sim$. Thus, $S/\sim$ is an abelian n.p.o. semigroup. To see that $S/\sim$ is also implicative, we show that the implicative operation $*$ defined on S is hereditary on $S/\sim$. In fact, $*$ is well-defined on $S/\sim$. For if $x \sim x'$ and $y \sim y'$ on S, then there exists elements c and d such that $cx \leq x'$, $cx' \leq x$, $dy \leq y'$ and $dy' \leq y$. For the sake of brevity, we denote $x * y$ by u and $x' * y'$ by t. Then $ux \leq y$ implies that $(cd)ux' = du(cx') \leq (du)x = d(ux) \leq dy \leq y'$. This means that $(cd)(x * y) \leq x' * y'$. Similarly, $tx' \leq y'$ implies that $(cd)tx = dt(cx) \leq dtx' \leq dy' \leq y'$. This means that $(cd)(x' * y') \leq x * y$. Thus, we have show that $x * y \sim x' * y'$ and hence $x' * y'/\sim (=) x * y/\sim$. To see $*$ is an implication on $S/\sim$, we still have to show that $z/\sim \cdot x/\sim (\leq) y/\sim$ iff $z/\sim (\leq) x * y/\sim$. For any $x' \in x/\sim$, $y' \in y/\sim$ and $z' \in z/\sim$, there exists an element c in J such that $c(z'x') \leq y'$. This implies that $c' \leq x' * y'$ and so $z'/\sim (\leq) x' * y'/\sim (=) x * y/\sim$.

Conversely, suppose $z/\sim \ (\leq) \ x * y/\sim$. Then for any $z' \in z/\sim$, $t \in x * y/\sim$, there exists an element c in J such that $cz' \leq t$. Also, for any $x' \in x/\sim$ and $y' \in y/\sim$, there exists d in J such that $dt \leq x' * y'$, $d(x' * y') \leq t$. These facts imply $dtx' \leq y'$. As $(cd)x'z' = dx'(cz') \leq dx't \leq y'$ and $cd \in J$. Hence $xz/\sim \ (\leq) \ y/\sim$. Because $xz/\sim \ (=) \ x/\sim \cdot z/\sim$, so $z/\sim \cdot x/\sim \ (\leq) \ y/\sim$. The proof is completed.

In view of Lemma 3.2, we consider the mapping $\alpha : (S, \cdot, \leq, *) \longrightarrow (S/\sim , \cdot, (\leq), *)$ defined by $x \longmapsto x/\sim$. It can be easily verified that α is an onto mapping and $\alpha(x * y) \ (=) \ x * y/\sim \ (=) \ x/\sim * y/\sim \ (=) \ \alpha(x) * \alpha(y)$. Thus α is indeed an implicative homomorphism from $(S, \cdot, \leq, *)$ onto $(S/\sim , \cdot, (\leq), *)$. We call α to be the canonical implicative homomorphism of S .

Parallel to the usual homomorphism theorem in algebraic systems, we obtain the following homomorphism theorem for implicative n.p.o. abelian semigroups.

<u>Theorem 3.3</u> (Homomorphism Theorem). Let α be an implicative homomorphism from an implicative n.p.o. abelian semigroup $(S, \cdot, \leq, *)$ onto an abelian implication n.p.o. semigroup $(S', \cdot, \leq, *)$, and let β be a canonical homomorphism from $(S, \cdot, \leq, *)$ onto $(S/\sim , \cdot, (\leq), *)$. with $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha$. Then there exists an implicative homomorphism Γ from $(S/\sim , \cdot, (\leq), *)$ onto $(S', \cdot, \leq, *)$ such that the following diagram is commutative.

$$\begin{tikzcd}
(S, \cdot, \leq, *) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & & (S', \cdot, \leq, *) \\
& (S/\ , \cdot, (\leq), *) \arrow[ur, dashed, "\Gamma"'] &
\end{tikzcd}$$

Moreover, if $\ker\beta = \alpha^{-1}(1')$, then Γ is an implicative isomorphism, that is, $(S/\sim, \cdot, (\leq), *) \cong (S', \cdot, \leq, *)$.

Proof: Define $\Gamma(x/\sim) = \alpha(x)$. It is not difficult to see that Γ is the required mapping. So, we only need to verify that Γ is injective, that is, we need to prove that for any $x, y \in S$, the following conditions are equivalent:

(i) $\alpha(x) = \alpha(y)$

(ii) Both $x * y$ and $y * x$ are in $\operatorname{Ker}\beta$.

(iii) $x/\sim \ (=)\ y/\sim$ in $S/\sim$.

As $\alpha(x) = \alpha(y)$ implies $\alpha(x * y) = \alpha(x) * \alpha(y) = \alpha(x) * \alpha(x) = 1'$. Hence $x * y \in \alpha^{-1}(1') = \operatorname{Ker}\beta$. Similarly, $y * x \in \operatorname{Ker}\beta$. Thus (i) $\Rightarrow$ (ii). Denote $x * y$ by c, $y * x$ by d. By the definition of $*$, we know that there exists elements c and d in s such that $cx \leq y$ and $dy \leq x$. Hence $(cd)x \leq dy \leq y$ and $(cd)y \leq cx \leq x$. As $cd \in J$, so $x \sim y$. This means that $x/\sim \ (=)\ y/\sim$ in $S/\sim$. Thus (ii) $\Rightarrow$ (iii). For (iii) $\Rightarrow$ (i), we suppose c is an element in $\operatorname{Ker}\beta = \alpha^{-1}(1')$ such that $cx \leq y$, $cy \leq x$. Then we have $\alpha(cx) = \alpha(c)\,\alpha(x) \leq \alpha(y)$ and $\alpha(cy) = \alpha(c)\,\alpha(y) \leq \alpha(x)$. Because $\alpha(c) = 1'$, therefore we have $\alpha(x) \leq \alpha(y)$ and $\alpha(y) \leq \alpha(x)$. Hence $\alpha(x) = \alpha(y)$. This shows that Γ is an isomorphism.

In closing this section, we construct an example to illustrate the above theorem. In fact, we illustrate that there exists a 1-1 correspondence between the order filters and the homomorphic images of an implicative n.p.o. abelian semigroup. The construction is technical.

Example 3.4 Let $(Z^+, \cdot)$ (Z^+ is the set of positive integers) be a semigroup with usual multiplication. For all $a, b \in (Z^+, \cdot)$, define $a \leq b$ if and only if $b \mid a$. Then $(Z^+, \cdot, \leq)$ is an abelian n.p.o.

semigroup with the greatest element 1. Let $d = (x, y)$, the H.C.F. of x and y in Z^+. By the definition of $\leq$, we know that $zx \leq y$ if and only if $y|zx$ for all $z \in Z^+$. This implies that $\frac{y}{d} | z(\frac{x}{d})$. Because $(\frac{x}{d}, \frac{y}{d}) = 1$, so $(\frac{y}{d}) | z$. Therefore $z \leq \frac{y}{d}$. Conversely, if $z \leq \frac{y}{d}$, then $\frac{y}{d} | z$ and $y | dz$. Hence $y|xz$ and so $xz \leq y$. In other words, we have shown that $z \leq \frac{y}{d}$ if and only if $zx \leq y$. thus, an implication $*$ is defined on $(Z^+, \cdot, \leq)$ by $x*y = \frac{y}{d} \in Z^+$ with $d = (x, y)$. Thus $(Z^+, \cdot, \leq, *)$ is an implicative n.p.o. semigroup.

Let J be an order filter of $(Z^+, \cdot, \leq, *)$. For any element $z \neq 1$ in Z^+, we have $z = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where the p_i's are prime numbers and α_i's are positive integers. If $z \in J$ then $p_i | a$ for all $i = 1, 2, \ldots, k$. Hence p_i belongs to J for all $i = 1, 2, \ldots, k$. Consider the set $P(J) = \{ p \in Z^+ : p$ is prime and $p \in J \}$. Then $J = \{ z \in Z^+ : z = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some $k \in \mathbb{N}$ with $p_i \in P(J)$ for $i = 1, 2, \ldots, k$ and each α_i is a non-negative integer $\}$. For instance, the sets $\{ 2^n : n \geq 0\}$, $\{2^n \cdot 3^m : n, m \geq 0 \}$ are order filters of $(Z^+, \leq)$. It should be noted that 1 always belongs to J. As there are infinite many number of primes in Z^+, there are infinite many number of order filters in $(Z^+, \cdot, \leq, *)$. Apply Theorem 3.3, there is a 1-1 correspondence between order filters and implicative homomorphic images (up to isomorphism) of $(Z^+, \cdot, \leq, *)$. Thus, there are infinite many number of implicative homomorphisms defined on Z^+. Furthermore, for any order filter J_1 of S, let p be a prime number such that $p \notin J$ and let J_2 be the order filter generated by p and J_1, then $J_1 \subsetneq J_2$. This shows that $(Z^+, \cdot, \leq, *)$ does not has ultra order filters. On the contrary, if $(Z^+, \cdot, \leq, *)$ has infinite many number of minimal order filters, then for any order filter J_1 of S we let p be any prime belonging to J_1 . Thus $J_p = \{ p^k : k \in \mathbb{N} \}$ is a minimal order filter of S . It can be observe that any minimal order filter of $(Z^+, \cdot, \leq, *)$ must be of this form. Hence the corresponding homomorphic images of $J_p = \{ p^k : k \in \mathbb{N} \}$ are precisely the set $Z_p^+ = \{ [x] : x, y \in [x]$ if and only if $x = p^k y$ for some integer $k \}$ up to isomorphism .

By this example, we conclude that the minimal order filters of $(Z^+, \cdot, \leq, *)$ are precisely those order filters J_p's with p a prime number such that $J_p = \{ p^k : k \in \mathbb{N} \}$. Thus, all the maximal implicative homomorphic images of $(Z^+, \cdot, \leq, *)$ are precisely those sets Z_p^+ 's, as defined above.

Chapter III

Pseudo-complemented implicative abelian semigroups

§0. Introduction

The purpose of this chapter, is to study the concept of pseudo-complement in lattices to p.o. semigroups. This concept has already been investigated by O. Frink [6] in meet semilattices, who has shown that the greater part of the results concerning pseudo-complements in lattices may be derived without assuming the existence of the join operation. Later on, T.S. Blyth [3] generalized O. Frink's results to p.o. semigroups by considering the concept of pseudo-residuals. Further investigations of pseudo-complemented p.o. semigroups to be Glivenko semigroup and Browverian semigroups were due to M.F. Jonowitz and C.S. Johnson, Jr. in [13].

In this chapter, we shall study the homomorphisms of pseudo-complemented implicative semigroups. An implicative homomorphism theorem for bounded implicative semilattices is obtained.

§1. Preliminaries

By a p.o. semigroup we shall mean a set S of elements on which is defined a closed binary associative multiplication and a partial ordering with respect to which the multiplication is isotone (that is, $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $z \in S$) .

By the zero element of a semigroup S we shall mean an element

0 with the property that $0x = x0 = 0$ for all x S . It is trivial to see that such an element is unique whenever it exists.

Let S be an implicative abelian n.p.o. semigroup with 0 . The maximal element contained in the annihilator $0 : a = \{x \in S : ax = 0\}$ if exists is called the pseudo-complement of a , and is denoted by a^* . In other words a^* is the element of S such that $aa^* = 0$ and if $ax = 0$, then $x \leq a^*$. We also denote a^* by $a * 0$, this is to emphasis that a^* is the greatest element to annihilate the element a of S . We use the symbol $a * b$ to denote the relatively pseudo-residuals of a by b , that is, $a * b$ is the greatest element in the set $\{y \in S : ay \leq b\}$. The reader should note that the $*$ defined here coincides with the implication in Chapter II. The p.o. semigroup S is said to be pseudo-complemented if every element of S is pseudo-complemented. We denote the pseudo-complemented implicative semigroup by $(S, \cdot, \leq, *, 0)$. Without ambiguity, we call S a pseudo-complemented implicative semigroup. The reader is reminded that the semigroup S is always partially ordered and that $x^{*} * y$ should be read as $(x * 0) * y$.

The followings are the basic properties of pseudo-complemented implicative semigroups.

<u>Lemma 1.1</u> Let $(S, \cdot, \leq, *, 0)$ be a pseudo-complemented abelian implicative semigroup with 1 and x, y are arbitrary elements of $(S, \cdot, \leq, *, 0)$. Then the following statements hold in $(S, \cdot, \leq, *, 0)$.

(1) $1^* = 0$; $x^* = 0$ if and only if $x = 0$.

(2) $x \leq y$ implies $x^* \geq y^*$ and $x^{**} \leq y^{**}$.

(3) $xx^* = 0$ and for any t S such that $xt = 0$ then $t \leq x^*$.

(4) $x \leq x^{**}$ and $x^* = x^{***}$.

(5) $(xy)^* = x * y^* = y * x^*$.

(6) $(x^{**} * y^*)^{**} = x^{**} * y^* = (x^*y)^*$.

(7) $x * y \leq y^{**} * x^*$; hence $(x * y)^{**} \leq y^{**} * x^*$.

(8) $(xy)^{**} = (x^{**}y)^{**} = (x^{**}y^{**})^{**}$; hence $x^{**}y^{**} \leq (xy)^{**}$.

(9) $x^{**} * y^{**} = y^{**} * x^*$.

(10) $y \leq y^{**} * x$, $y^* \leq y * x$.

(11) $(x * y)^{**} \leq x^{**} * y^* = y^{**} * x^*$.

(12) $(x^2)^{**} = x * x^*$; $(x^n)^* = x^{n-1} * x^*$ for all $n \geq 2$.

Proof: The above properties are well-known in semilattices. Of course, they can be proved verbatim in p.o. semigroup. For the sake of completeness, we provide the proofs.

(1) $x \cdot 1 \leq 0$ if and only if $x = 0$.

(2) Trivial.

(3) This follows directly from the definition of $*$.

(4) $x * x^{**} = x * (x^* * 0) = (xx^*) * 0 = 0 * 0 = 1$. Hence $x \leq x^{**}$. Subsitute x^* for x in $x \leq x^{**}$. Then $x^* \leq x^{***}$. By (2), $x \leq x^{**}$ implies that $x^* = x * 0 \geq x^{**} * 0 = x^{***}$. Therefore, $x^* = x^{***}$.

(5) $(xy)^* = (xy) * 0 = x * (y * 0) = x * y^*$. Similarly $(xy)^* = (yx)^* = y * x^*$.

(6) $x^{**} * y^* = x^{**} * (y * 0) = (x^*y) * 0 = (x^*y)^*$. By (4), $(x^*y)^* = (x^*y)^{***} = (x^{**} * y^*)^{**}$. This implies that $x^{**} * y^* = (x^{**} * y^*)^{**}$.

(7) $y \leq y^{**}$ implies that $x * y \leq x * y^{**} = x * (y^* * 0) = (xy^*)^* = y^{**} * x^*$ (By (6)).

(8) By (4) and (5), $(x^{**}y^{**})^{**} = (x^{**} * y^*)^* = [x^{**} * (y*0)]^* = (x^{**}y)^{**}$. By (5), $(x^{**}y)^{**} = (y * x^*)^* = [y * (x * 0)]^* = (yx)^{**}$. Similarly $(x^{**}y^{**})^{**} = (xy^{**})^{**}$.

(9) $x^{**} * y^{**} = x^{**} * (y^* * 0) = (y^*x^{**})^* = y^{**} * x^*$.

(10) $y^*y = yy^* = 0 \leq x$. So $y \leq y * x$, $y^* \leq y * x$.

(11) By (7), $x * y \leq y^* * x^* = x^{***} * y^{**}$. Therefore by (6), $(x * y)^{**} \leq (x^{***} * y^{**})^{**} = x^{***} * y^{**}$.

(12) The result follows from (5).

§2. The Construction of a Pseudo-complemented implicative semigroup

Definition 2.1 An element x of an pseudo-complemented abelian semigroup S is said to be closed if and only if $x^{**} = x$.

This definition follows from W.C. Nemitz in [15].

Lemma 2.2 Let S be a pseudo-complemented implicative abelian semigroup. Then the following properties hold:

(i) $x^{**} = x$ if and only if $x = y^*$ for some y S .

(ii) Let $S^{**} = \{ x : x$ is closed $\}$. Then $S^{**} = \{ y^* : y \in S \}$ and clearly 0 and 1 are both in S^{**} .

(iii) Let $x \in S^{**}$, $y \in S^{**}$. Then $x * y = \max \{ z \in S : xz \leq y \}$ is also an element of S^{**} . In fact, $x * y = y^* * x^*$.

Proof: These properties are easy to prove and their proofs are hence omitted.

Let S^{**} be the set of all closed elements of S . Clearly S^{**} is a p.o. subset of S with the induced partial ordering "$\leq$" inherited from $(S, \cdot, \leq)$. Define a binary operation "o" on $S^{**} \times S^{**} \longrightarrow S^{**}$ defined by $a \circ b = (ab)^{**}$. We now show that a pseudo-complemented implicative abelian semigroup can be constructed under the binary operation "o" .

Theorem 2.3 The set of closed elements of a pseudo-complemented implicative abelian semigroup under the binary operation "o" is a pseudo-complemented implicative abelian semigroup.

Proof: Let a, b be elements of S^{**}. Clearly $(ab)^{**} \in S^{**}$. For any elements $a, b, c \in S^{**}$, we have $a \circ (b \circ c) = a \circ (bc)^{**} = [a(bc)^{**}]^{**}$. By Lemma 1.1 (8), we know that $[a(bc)^{**}]^{**} = [a(bc)]^{**} = [(ab)c]^{**} = [(ab)^{**}c]^{**} = (ab)^{**} \circ c = (a \circ b) \circ c$. Thus $a \circ (b \circ c) = (a \circ b) \circ c$ and so "o" is an associative binary operation on S^{**}. It is trivial to see that $a \circ b = b \circ a$, $1 \circ a = a \circ 1 = a$ and $0 \circ a = a \circ 0 = 0$. Hence $(S^{**}, \circ)$ is an abelian semigroup with 0 and 1. Obviously, $(S^{**}, \circ, \leq)$ is p.o. semigroup with "o" inherited from S. To see that $(S^{**}, \circ, \leq)$ is negatively ordered, we observe that the following properties hold for any $a, b, c \in S^{**}$.

(i) $a \leq b$ implies $c \circ a = (ca)^{**} \leq (cb)^{**} = c \circ b$.

(ii) Similarly $a \circ c \leq b \circ c$.

(iii) $(a \circ b) = (ab)^{**} \leq a^{**} = a$.

(iv) Similarly $a \circ b \leq b$.

Thus, $(S^{**}, \circ, \leq)$ is a n.p.o. abelian semigroup with 0 and 1. Furthermore, by Lemma 2.2 (iii), we have $a * b \in S^{**}$ for any $a, b \in S^{**}$. Hence, for any element $c \in S^{**}$, $c \leq a \circ b$ implies $ca \leq b$ and $(ca) \leq (ac)^{**} \leq b$. These imply that $c * (a * b) = (ca)^{**} * b = (ca) * b = 1$ and so $c \leq a * b$. Consequently, $c \leq a * b$ if and only if $c \circ a \leq b$. Thus $(S^{**}, \circ, \leq, *, 0)$ is an implicative semigroup. Clearly, for any element a, b in S^{**}, we have $a \circ a^* = (aa^*)^{**} = 0^{**} = 0$ and $a \circ c = 0$ implies $(ac)^{**} = 0$, $(ac)^* = 1$. Thus, $c * a^* = c * (a * 0) = (ca)^* = 1$ and $c \leq a^*$. So, a^* is the pseudocomplement of a. Hence, $(S^{**}, \circ, \leq, *, 0, 1)$ is proved to be a pseudocomplemented implicative abelian semigroup.

Corollary 2.4 Let $\alpha : (S, \cdot, \leq, *, 0, 1) \longrightarrow (S^{**}, o, \leq, 0, 1)$ be such that $x \longmapsto x^{**}$. Then α is a surjective semigroup homomorphism.

Proof: Obviously, α is a well-defined mapping and is surjective. As $\alpha(xy) = (xy)^{**} = (x^{**}y^{**})^{**} = x^{**} \circ y^{**} = \alpha(x) \circ \alpha(y)$. Therefore α is a surjective semigroup homomorphism.

In general, the subset S^{**} of S need not be a semigroup under the semigroup multiplication of S . In particular, $(x * y)^{**} \neq x^{**} * y^{**}$. The following example is given to illustrate this situation:

Example 2.5 Let $S = \{1, a, b, c, 0\}$ be an ordered set with Cayley table and Hasse diagram as follows:

$\cdot$	1	a	b	c	0
1	1	a	b	c	0
a	a	c	0	0	0
b	b	0	0	0	0
c	c	0	0	0	0
0	0	0	0	0	0

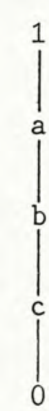


Table 1

Clearly $(S, \leq, \cdot)$ is an abelian n.p.o. semigroup.

As shown in the following table, $x * y$ exists for any x and y in S .

$*$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	a	a	b
b	1	1	1	a	a
c	1	1	1	1	a
0	1	1	1	1	1

Table 2

Thus $(S, \leq, \cdot, *, 0)$ is a pseudo-complemented implicative semigroup with $1* = 0$, $a* = b$, $b* = a$, $c* = a$ and $0* = 1$. Hence we have $1** = 1$, $a** = a$, $b** = b$, $c** = c$ and $0** = 0$. Therefore, $S** = \{1, a, b, 0\}$. As $a \cdot a = c \notin S**$, $S**$ does not form a semigroup under the semigroup multiplication of S. Moreover, $(b * c)** = a** = a$ while $b** * c** = b * b = 1$. Therefore, $(b * c)** \neq b** * c**$.

§3. The homomorphism Theorem

Recall that an implicative semilattice L is a system $(L, \leq, \wedge, *)$ in which L is a non-empty set, $\leq$ is a partial order on L, $\wedge$ is a greatest lower bound with respect to $\leq$, and $*$ is a binary operation in L such that for any x, y and z of L, $z \leq x * y$ if and only if $z \wedge x \leq y$. According to W.C. Nemitz [15], an implicative semilattice L is bounded if and only if L contains 0 such that $0 \neq 1$, and $x \geq 0$ for every $x \in L$. In this case, for $x \in L$, let $x* = 0 * x$. Then $x*$ plays the role of pseudo-complement of x in the sense of O. Frink [6]. Therefore bounded implicative semilattices and pseudo-complemented abelian n.p.o. semigroups. Conversely, if the multiplication "$\cdot$" of a pseudo-complemented n.p.o. semigroup is defined to be $x \cdot y = \inf\{x, y\} = x \wedge y$ for any x, y in S, then this pseudo-complemented abelian n.p.o. semigroup is a bounded implicative semilattice.

In this section, we shall show that the $\wedge$-semilattice homomorphism $\alpha : (L, \wedge, \leq, *, 0, 1) \longrightarrow (L**, \wedge, \leq, *, 0, 1)$ such that $x \longmapsto x**$ is a surjective implicative homomorphism.

<u>Definition 3.1</u> Let $(S, \cdot, \leq, *, 0)$ and $(S', \cdot, \leq, *, 0)$ be two implicative abelian n.p.o. semigroups. If $\alpha : (S, \cdot, \leq, *, 0) \longrightarrow$

$(S', \cdot, \leq, *, 0)$ such that $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all $x, y \in S$, then α is called an <u>implicative homomorphism</u>.

It should be noticed that implicative homomorphisms between n.p.o. semigroups are different from semigroup homomorphisms. The following is an example:

<u>Example 3.2</u> In Example 2.5, $S = \{1, a, b, c, 0\}$ with multiplication as shown in Table 1 and the partial order $\leq$ is such that $1 \geq a \geq b \geq c \geq 0$. From Table 2, $1* = 0$, $a* = b$, $b* = a$, $c* = a$ and $0* = 1$. Hence $1** = 1$, $a** = a$, $b** = b$, $c** = b$ amd $0** = 0$. Therefore $S** = \{1, a, b, 0\}$ and $(S**, \leq)$ forms a p.o. set with the partial order "$\leq$" induced by $(S, \leq)$. On $S**$, define the multiplication "o" as shown in Table 3 below:

o	1	a	b	0
1	1	a	b	0
a	a	b	0	0
b	b	0	b	0
0	0	0	0	0

Table 3.

From Table 3, we see that for any elements $x, y \in S**$, $x \circ y = (xy)**$. For instance, $a \circ a = b = c** = (a \cdot a)**$ and $(a \circ b) = 0 = 0** = (ab)**$. As it has been shown in Theorem 2.3, $(S**, o, \leq, *, 0, 1)$ forms a pseudo-complemented implicative abelian semigroup with the implication operation "$*$" given in Table 4 below:

*	1	a	b	0
1	1	a	b	0
a	1	1	a	b
b	1	1	1	a
0	1	1	1	1

Let $\alpha : (S, \cdot, \leq, *, 0, 1) \longrightarrow (S^{**}, o, \leq, *, 0, 1)$ such that $\alpha(x) = x^{**}$ for all $x \in S$. Then $\alpha(1) = 1$, $\alpha(a) = a$, $\alpha(b) = b$, $\alpha(c) = b$ and $\alpha(0) = 0$.

By Corollary 2.4, α is a surjective semigroup homomorphism. In Table 1, $b, c \in S$ and $b * c = a$ in S . Hence $\alpha(b * c) = \alpha(a) = a$. But in S^{**} , $\alpha(b) * \alpha(c) = b * b = 1$. So $\alpha(b * c) \neq \alpha(b) * \alpha(c)$. Therefore, α is not an implicative homomorphism.

<u>Lemma 3.3</u> Let $(S, \cdot, \leq, *, 0)$ be a pseudo-complemented $\wedge$-semilattice, that is, for every $a, b \in S$, $a \cdot b = a \wedge b = \inf \{a, b\}$. Then the binary operation "o" defined by $a \circ b = (ab)^{**}$ in S^{**} is compatible with the semigroup multiplication "•" , that is $a \circ b = a \cdot b$ for all $a, b \in S^{**}$.

Proof: For any elements a and b in S^{**} , $a \circ b = (a \cdot b)^{**} = (a \wedge b)^{**}$. $a \wedge b \leq a$ and $a \wedge b \leq b$ imply that $(a \wedge b)^{**} \leq a^{**} = a$ and $(a \wedge b)^{**} \leq b^{**} = b$. So $(a \wedge b)^{**} \leq a \wedge b$. So $a \wedge b \leq (a \wedge b)^{**}$, we conclude that $a \wedge b = (a \wedge b)^{**}$. Therefore, $a \circ b = a \wedge b = a \cdot b$.

<u>Lemma 3.4</u> For any elements x and y in a bounded implicative $\wedge$-semilattice $(L, \wedge, \leq, *, 0, 1)$, $(x * y)^{**} = x^{**} * y^{**}$.

Proof: Let $u = x * y$, then $u \wedge x \leq y$ and $(u \wedge x)^{**} \leq y^{**}$. By Lemma 1.1(8) and Lemma 3.3, $(u \wedge x)^{**} = (u^{**} \wedge x^{**})^{**} = u^{**} \wedge x^{**} \leq y^{**}$. But $u^{**} \wedge x^{**} \leq y^{**}$ implies $u^{**} \leq x^{**} * y^{**}$. Hence $(x * y)^{**} \leq x^{**} * y^{**}$.

Conversely, by Lemma 1.1(8), (9), $(x^{***} * y^{**}) * (x * y)^{**} = (y^* * x^*) * (x * y)^{**} = (y^* \wedge x)^{**} * (x * y)^{**} = (y^* \wedge x^{**})^{**} * (x * y)^{**} = (y^* \wedge x^{**})^{**} * (x * y)^{**} = (x * y)^{**} * (y^* \wedge x^{**})$. By Chapter II, Theorem 1.5(6), $(x * y)^{**} * (y^* \wedge x^{**}) = [(x * y)^{**} * y^*] \wedge [(x * y)^{**} * x^{**}]$. By chapter II, Theorem 1.5(3), $(x * y) \geq y$. So $(x * y)^* \leq y^*$ and $(x * y)^{**} * y^* = 1$. Hence $(x * y)^{**} * (y^* \wedge x^{**}) = (x * y)^{**} * (x^*)^* = (x^*)^{***} * (x * y)^{**} = x^* * (x * y)^{**}$. By Theorem 1.4(4)(b), $x^* * (x * y)^{**} \geq x^* * (x * y) = (x^* x)y = 0 * y = 1$. Therefore $(x^{***} * y^{**}) * (x * y)^{**} = 1$ and $x^{***} * y^{**} \leq (x * y)^{**}$. We conclude that $x^{***} * y^{**} = (x * y)^{**}$.

<u>Theorem 3.5</u> The $\wedge$-semilattice homomorphism $\alpha : (L, \wedge, \leq, *, 0, 1) \longrightarrow (L^{**}, \wedge, \leq, *, 0, 1)$ such that $x \longmapsto x^{**}$ is a surjective implicative homomorphism.

Proof: By Chapter II, Theorem 2.2 and Chapter III Corollary 2.4, it suffices to prove that $\alpha(x * y) = \alpha(x) * \alpha(y)$. By Lemma 3.4, $\alpha(x * y) = (x * y)^{**} = x^{***} y^{**} = \alpha(x) * \alpha(y)$. Therefore, α is a surjective implicative homomorphism.

Chapter IV

Construction of Pesudo-indexed semigroups

§0. Introduction

The theory of semilattice decomposition of a semigroup has been developed by M. Petrich [16]. Naturally, one would consider the opposite, that is, the theory of semilattice composition of semigroups. The study of such composition theory was initiated by R. Yoshida and M. Yamada [24]. By using the ideal extension theory developed by M. Petrich and P.O. Grillet [9]; and a construction theorem for arbitrary semigroups was established. this theorem was stated and proved by M. Petrich in his recent text ([16]; p.94).

In this chapter, we study the semilattice composition of a particular class of semigroups, namely the pseudo-index semigroups. A construction theorem similar to the one given by M. Petrich in [16] is obtained. It is worth to point out that in our construction, the theory of extension is not essentially required. Although M. Petrich detailed with the general case, our construction is more intrinsic.

§1. Notation and construction

The following is called the construction problem in the theory of semilattice composition: Given a semilattice Y and a collection of pairwise disjoint semigroups S_α indexed by Y, construct a semigroup S which admitsa homomorphism φ onto Y and for which $\varphi^{-1} = S_\alpha$

for all $\alpha \in Y$. State differently, S can be taken to be the disjoint union of all S_α and must have a multiplication for which $S_\alpha S_\beta \subset S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

The reader should be aware that such an S may not exist! However, if such S does exist, then S is called a semilattice Y of semigroups S_α .

The following definition is formulated by M. Petrich in [16] and by R. Yoshida and M. Yamada in [24].

<u>Definition 1.1</u> A semigroup S is a semilattice Y of semigroups S_α if there exists a homomorphism φ of S onto the semilattice Y such that $S_\alpha = \varphi^{-1}\alpha$ for all $\alpha \in Y$.

Following the definition and ideas, we shall construct a particular class of semilattices Y , namely the <u>pseudo-indexed</u> semilattices in which the elements are taken from the given semigroup S . Weshall show that the semigroup S with such semilattice Y is a semilattice Y of semigroups S_α , for all $\alpha \in Y$.

<u>Construction 1.2</u> Let C be a non-empty subset of an abelian semigroup $(S, \cdot)$. Define a partial ordering "$\leq$" on C such that for any elements a, b in C , inf a, b exists. Denote inf $\{a, b\}$ by $a \wedge b$. Thus $(C, \wedge)$ becomes a semilattice under the partial ordering "$\leq$" . [Note that "$\leq$" is only defined on C , but not on the whole semigroup S . That is, S need not be a semilattice. Remind that C is only a subset of S , but the multiplication on C differs from S] .

Let $* : (S, \cdot) \longrightarrow (C, \wedge)$ be a surjective mapping such that

$*(x) = x^*$ for all $x \in S$. "$*$" is called a pseudo-semigroup homomorphism if the following conditions are satisfied for any element x in S :

(I) $x^{***} = x^*$.

(II) $(xy)^* = x^* \wedge y^*$.

(III) $(xy)^{**} = x^{**}y^{**}$.

(IV) $xx^{**} = x$.

[Note: the properties (I) - (IV) are similar to the pseudo-complemented properties in semilattices, that is why we call "$*$" to be the pseudo-semigroup homomorphism].

Definition 1.3 A semilattice $(C, \wedge)$ together with a pseudo-semigroup homomorphism "$*$" from $(S, \cdot) \longrightarrow (C, \wedge)$ is called a Pseuo-indexed semilattice.

The following shows that examples of pseudo-indexed semilattices can be constructed.

Example 1.4 Let $S = \{a, b, c\}$ with Cayley table:

$\cdot$	a	b	c
a	a	b	a
b	b	b	b
c	a	b	c

Then it is easy to verify that $(S, \cdot)$ is an abelian semigroup.

Let $C = \{c, b\} \subsetneq S$ with $b \leq c$. Then $(C, \wedge)$ is a semilattice.

Define $* : (S, \cdot) \longrightarrow (C, \wedge)$ by the following table

$S \backslash C$	$*$	$**$	$***$
a	c	c	c
b	b	b	b
c	c	c	c

Then for any element $x \in S$, we have

(i) $x*** = x*$ (as shown in the above diagram)

(ii) $(xb)* = b* = b$ and $x* \wedge b* = b* = b$. Hence $(xb)* = x* \wedge b*$.

Also $(ac)* = a* = c = a* \wedge c*$, $(a \cdot a)* = a* = c = a* \wedge a*$ and $(cc)* = c* = c* \wedge c*$.

(iii) $(xb)* = b** = b = x**b**$, $(aa)** = (a* \wedge a*)* = c* = c = a**a**$.

Similarly, $(ac)* = a**c**$ and $(cc)** = c**c**$.

(iv) $aa** = ac = a$, $bb** = b \cdot b = b$ and $cc** = c \cdot c = c$.

Thus, $(C, \wedge)$ is indeed a pseudo-indexed semilattice.

§2. Properties of pseudo-indexed semilattices and semigroups

The following theorem includes a number of the basic properties of a pseudo-indexed semilattice.

Theorem 2.1 Let $(C, \wedge)$ be a semilattice together with a pseudo-semigroup homomorphism "$*$" from $(S, \cdot) \longrightarrow (C, \wedge)$. Then the following properties hold:

(1) For any element x in S, $(x^2)* = x*$. Inductively, $(x^n)* = x*$ for all natural number n. Similarly, $(x^n)** = x**$ for all n.

(2) For any element a in C, $a** = a$.

(3) If $a \leq b$ in C, then $a* \leq b*$, that is, $*$ is order-preserving in C.

(4) For any element a and b in C, $(a \wedge b)^* = a^* \cdot b^*$ in S.

(5) If $a \leq b$ in C, $a^*b^* = b^*a^* = a^*$.

(6) For any elements a and b in C, $(ab)^{**} = ab$, that is, $(C, \cdot)$ forms a subsemigroup of $(S, \cdot)$.

(7) Every element a in C is an idempotent and $(C, \cdot)$ is a semilattice.

(8) If x and y are elements in S with $x^* = y^* = a$ in C, then $(xy)^* = a$.

(9) If x is an element in S and $x^* = a$ in C, then $xa^* = x$.

Proof: (1) By (II), $(x^2)^* = x^* \wedge x^* = x^*$ for any element x in S.

(2) Now $a = x^*$ for some x in S. By (I), $x^{***} = x^*$. Hence $a^{**} = x^{***} = x^* = a$.

(3) Let $a = u^*$ and $b = v^*$ for some u and v in S. $a^* \wedge b^* = u^{**} \wedge v^{**} = (uv)^{**} = (u^* \wedge v^*)^* = (a \wedge b)^* = a^*$. Therefore, $a^* \leq b^*$ if $a \leq b$.

(4) Let $a = u^*$ and $b = v^*$ for some u and v in S. $(a \wedge b)^* = (u^* \wedge v^*)^*$. By (II) and (III), $(u^* \wedge v^*)^* = (uv)^{**} = u^{**}v^{**} = a^*b^*$.

(5) Let $a = u^*$ and $b = v^*$ for some u and v in S. $a^*b^* = u^{**}v^{**} = (uv)^{**} = (u^* \wedge v^*)^* = (a \wedge b)^* = a^*$ as $a \leq b$.

(6) By (III), $(ab)^{**} = a^{**}b^{**}$. Hence by (2), $(ab)^{**} = ab$.

(7) By (3), a^2 is an element of C. Therefore, $a^2 = (a^2)^{**} = (a^* \wedge a^*)^* = a^{**} = a$.

(8) Trivial

(9) By (IV).

<u>Lemma 2.2</u> Let $(S, \cdot)$ be a semigroup with a pseudo-indexed semilattice $(C, \wedge)$. For any $a \in C$, let $D_a = \{x \in S : x^* = a\}$, then the following properties hold:

(1) $(D_a, \cdot)$ is a subsemigroup of $(S, \cdot)$.

(2) For any $a, b \in (C, \wedge)$, $D_a \cdot D_b \subseteq D_{a \wedge b}$.

(3) For any $a, b \in (C, \wedge)$ with $a \leq b$, $D_a \cdot D_b \subset D_a$.

Proof: For any elements $a, b \in C$, let $x \in D_a$, $y \in D_b$. By (II), $(xy)^* = x^* \wedge y^* = a \wedge b$, so $xy \in D_{a \wedge b}$. Hence $D_a \cdot D_b \subseteq D_{a \wedge b}$, and (2) is proved. If $a \leq b$, then $a \wedge b = a$ and so $D_a \cdot D_b \subset D_a$. Therefore (3) is proved. When $a = b$, then $D_a \cdot D_a \subset D_{a \wedge a} = D_a$ and (1) is true.

In view of Lemma 2.2, we formulate the following definition which will be used later on.

Definition 2.3 Let $(S, \cdot)$ be a semigroup with pseudo-indexed semilattice $(C, \wedge)$. Then $D_a = \{ x \in S : x^* = a \}$ is called an pseudo-indexed semigroup.

§3. Construction theorem

The following theorem is a construction theorem which gives a special kind of semilattice of semigroup D_a ; it is reminiscent of a retract extension as cited by M. Petrich in ([16]; p.97).

Theorem 3.1 (Construction Theorem) Let $(C, \wedge)$ be a pseudo-indexed semilattice; for every $a \in (C, \wedge)$, let D_a be the pseudo-indexed semigroup on $(C, \wedge)$. For every pair $a, b \in (C, \wedge)$ such that $a \geq b$. Define $\psi_{a,b} : (D_{a}, \cdot) \to (D_{b}, \cdot)$ by $\psi_{a,b}(x) = xb^*$. Let $S = \bigcup_{a \in (C,\wedge)} D_a$ with $x \cdot y = x\psi_{a,b\wedge b}\, y\psi_{b,a\wedge b}$ $(x \in D_a, y \in D_b)$. Then $(S, \cdot)$ is a semilattice $(C, \wedge)$ of pseudo-indexed semigroups and is a subdirect product of semigroups S with zero adjoined.

In order to prove this theorem, we need to quote a result which appeared in the text of M. Petrich [16] (III 7.7 Proposition, p.97).

Lemma 3.2 Let Y be a semilattice; for every $\alpha \in Y$ let S_α be a semigroup and assume that the semigroups S_α are pairwise disjoint. For every pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\psi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ be a homomorphism such that $\psi_{\alpha,\alpha}$ is the identity mapping and $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ if $\alpha > \beta > \gamma$. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ with multiplication $a * b = \psi_{\alpha,\alpha\beta}(a)\, \psi_{\beta,\alpha\beta}(b)$. for $a \in S_\alpha$, $b \in S_\beta$. Then S is a semilattice Y of semigroups S_α and is a subdirect product of semigroups S_α with a zero possibly adjoined.

Thus, in proving Theorem 3.1, we only need to verify that the mapping $\psi_{a,b}$ defined in Theorem 3.1 satisfies all the conditions stated in Lemma 3.2. That is, we need to prove the following properties:

(i) For every $a, b \in (C, \wedge)$ such that $a \geq b$, $\psi_{a,b}$ is a semigroup homomorphism.

(ii) For every $x, y \in (S, \cdot)$, $x \cdot y = \psi_{a,a\wedge b}(x)\, \psi_{b,a\wedge b}(y)$.

(iii) For $a, b, c \in (C, \wedge)$ with $a \geq b \geq c$, $\psi_{a,a}$ is the identical semigroup homomorphism on $(D_a, \cdot)$.

(iv) For every $x \in D_a$, $\psi_{b,c}\, \psi_{a,b}(x) = \psi_{a,c}(x)$ where $a \geq b \geq c$.

(v) $\psi_{a,b}(a^*) = b^*$.

Proof: (i) For any element x in D_a, $(xb^*)^* = x^* \wedge b^{**} = a \wedge b = b$ and $xb^* \in D_b$. Hence $\psi_{a,b}$ is well-defined. For any elements x and y in D_a, $\psi_{a,b}(xy) = (xy)b^* = (xb^*)(yb^*) = \psi_{a,b}(x)\, \psi_{a,b}(y)$. i.e. ψ_{ab} is a subsemigroup homomorphism.

(ii) If x and y are elements of S with $x^* = a$, $y^* = b$ in C, then $xy \in D_{a\wedge b}$. Therefore, $xy = (xy)(xy)^{**} = (xy)(a \wedge b)^* = x(a \wedge b)^* y(a \wedge b)^* = \psi_{a,a\wedge b}(x)\, \psi_{b,a\wedge b}(y)$.

(iii) Trivial.

(iv) $\psi_{b,c}\psi_{a,b}(x) = \psi_{b,c}(xb^*) = (xb^*)c^* = x(b^*c^*)$. By Theorem 2.1(5), $b^*c^* = c^*$. Therefore, $\psi_{b,c}\,\psi_{a,b}(x) = xc^* = \psi_{a,c}(x)$.

(v) By Theorem 2.1(5), $\psi_{a,b}(a^*) = a^*b^* = b^*$.

<u>Definition 3.3</u> The semigroup $(S, \cdot)$ constructed in Theorem 3.1 is called a strong pseudo-indexed semilattice of pseudo-indexed semigroups D_a and will be denoted by $S = [C, D_a; \psi_{a,b}]$.

The following is an isomorphism theorem for strong pseudo-indexed semilattice of pseudo-indexed semigroups.

<u>Theorem 3.4</u> Let $S = [C, D_a, \psi_{a,b}]$ and $S' = [C', D'_a, \psi_{a',b'}]$ be strong pseudo-indexed semilattice C and C' of pseudo-indexed semigroups D_a and $D_{a'}$ respectively Let h be a semilattice isomorphism from $(C, \wedge)$ onto $(C', \wedge)$. For every element $a \in C$, let β_a be an isomorphism from $(D_a, \cdot)$ into $(D_{a'}, \cdot)$ where $a' = h(a)$. Moreover, for $a > b$ in C and any $x \in D_a$, assume $\beta_b(xb^*) = \beta_a(x)\,\beta_b(b^*)$. Then the mapping $\alpha : (S, \cdot) \longrightarrow (S', \cdot)$ such that $x \mapsto \beta_a(x)$ for any element $x \in D_a$, is an isomorphism from S onto S' .

Proof: We first claim that the following diagram is commutative, that is, $\psi_{a',b'}\,\beta_a = \beta_b\,\psi_{a,b}$:

$$\begin{tikzcd} D_a \arrow[r, "\beta_a"] \arrow[d, "\psi_{a,b}"'] \arrow[dr, phantom, "\circlearrowright"] & D_{a'} \arrow[d, "\psi_{a',b'}"] \\ D_b \arrow[r, "\beta_b"] & D_{b'} \end{tikzcd}$$

For any x in D_a, $\psi_{a',b'}\beta_a(x) = \beta_a(x)b'^*$. By Lemma 8(c), we have $b'^* = \beta_b(b^*)$. Hence $\psi_{a',b'}\beta_a(x) = \beta_a(x)\beta_b(b^*) = \beta_b(xb^*)$ $= \beta_b\psi_{a,b}(x)$. This shows that the diagram is commutative.

Clearly α is well-defined. For any elements $x, y \in S$, $x \in D_p$, $y \in D_q$ for some $p, q \in C$. Thus, we have shown that $xy \in D_{p\wedge q}$ and $\alpha(xy) = \beta_{p\wedge q}(xy)$. On the other hand, $\alpha(x)\alpha(y) =$ $\beta_p(x)\cdot\beta_q(y) = \psi_{p',p'\wedge q'}\beta_p(x)\psi_{q',p'\wedge q'}\beta_q(y)$. As the above diagram is commutative, $\psi_{p',p'\wedge q'}\beta_p(x) = \beta_{p\wedge q}\psi_{p,p\wedge q}(x)$ and $\psi_{q',p'\wedge q'}\beta_q = \beta_{p\wedge q}\psi_{q,p\wedge q}(y)$. These imply that $\alpha(x)\alpha(y) =$ $\beta_{p\wedge q}\psi_{p,p\wedge q}(x)\beta_{p\wedge q}\psi_{q,p\wedge q}(y) = \beta_{p\wedge q}[\psi_{p,p\wedge q}(x)\psi_{q,p\wedge q}(y)] = \beta_{p\wedge q}(xy)$. Therefore $\alpha(xy) = \alpha(x)\alpha(y)$ and so α is an homomorphism. Suppose x and y are elements of S such that $\alpha(x) = \alpha(y)$. Then we have $x \in D_a$, $y \in D_b$. Thus $\beta_a(x) = \beta_b(y)$ and $\beta_a(x) \in D_{a'}$, $\beta_b(y) \in D_{b'}$. These imply that $a' = b'$. Hence $a = b$ and $\beta_a(x) = \beta_a(y)$ implies $x = y$. So α is injective and clearly it is surjective.

Thus α is an isomorphism from S onto S'.

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